Date:- 27/05/2020

Time:- 10a.m. to 12p.m.

Chapter:- Hydrostatic

Topic:- Problems of Centre Of Pressure

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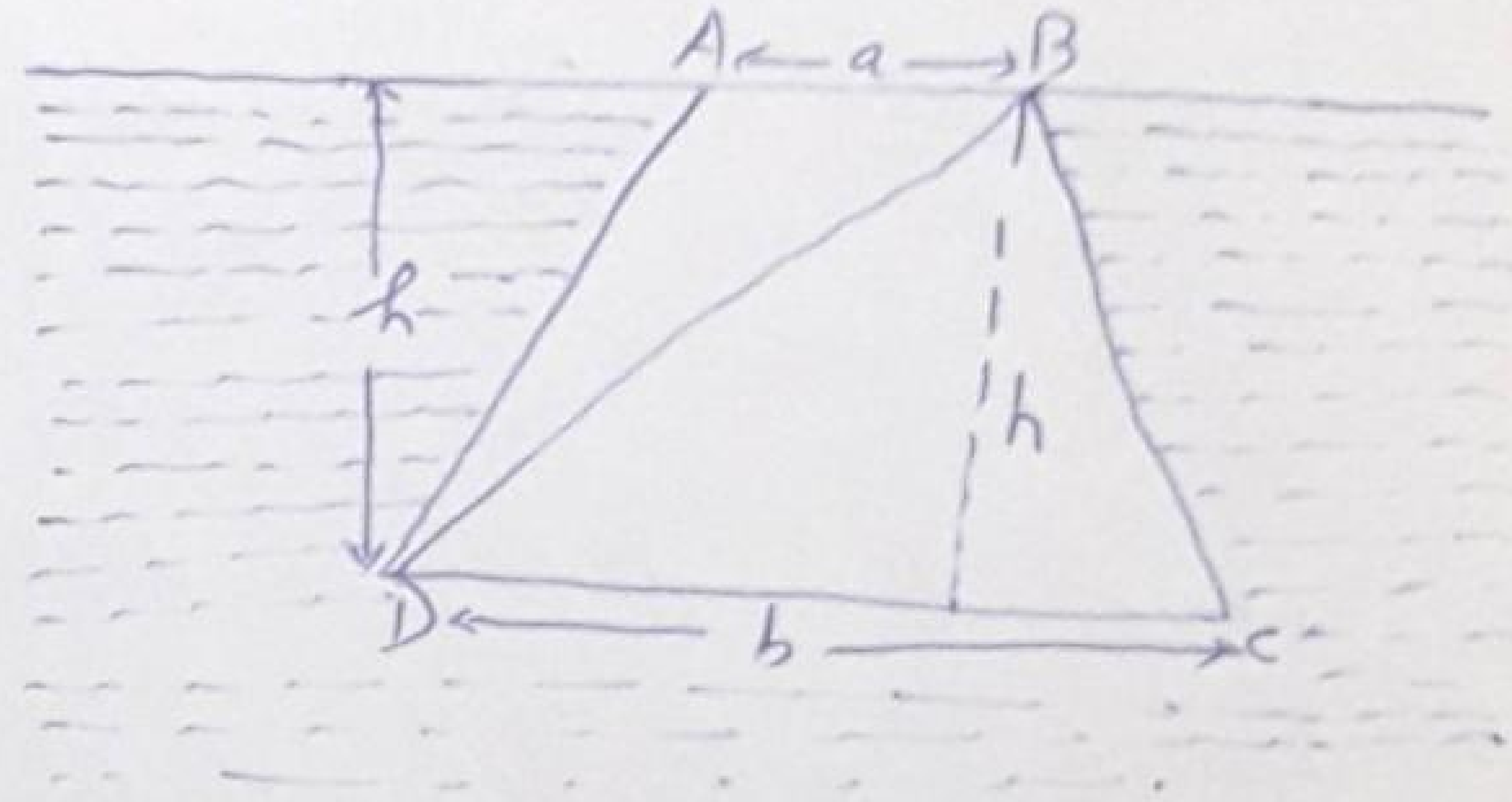
Problems of Centre of Pressure

Ex-1 Prove that the depth of the centre of pressure of a trapezium in a liquid with side a in the surface and the parallel side b at a depth h below the surface is $\dfrac{a+3b}{a+2b}\cdot\dfrac{h}{2}$

Sol^n:- Join the diagonal BD



Let ρ be the density of the liquid.

The thrust on $\triangle ABD = \rho g\cdot\frac{h}{3}\cdot\frac{1}{2}ah$
$= T_1$ (say)

and the depth of the C.P. of $\triangle ABD = \frac{h}{2} = Z_1$ (say).

Also the thrust on $\triangle BCD = \rho g\cdot\frac{2h}{3}\cdot\frac{1}{2}bh = T_2$ (say).

and the depth of the C.P of $\triangle BCD = \frac{3}{4}h = Z_2$ (say).

Trapezium $ABCD = \triangle ABD + \triangle BCD$

If $\bar{Z}$ be the required depth of the C.P of the trapezium ABCD.

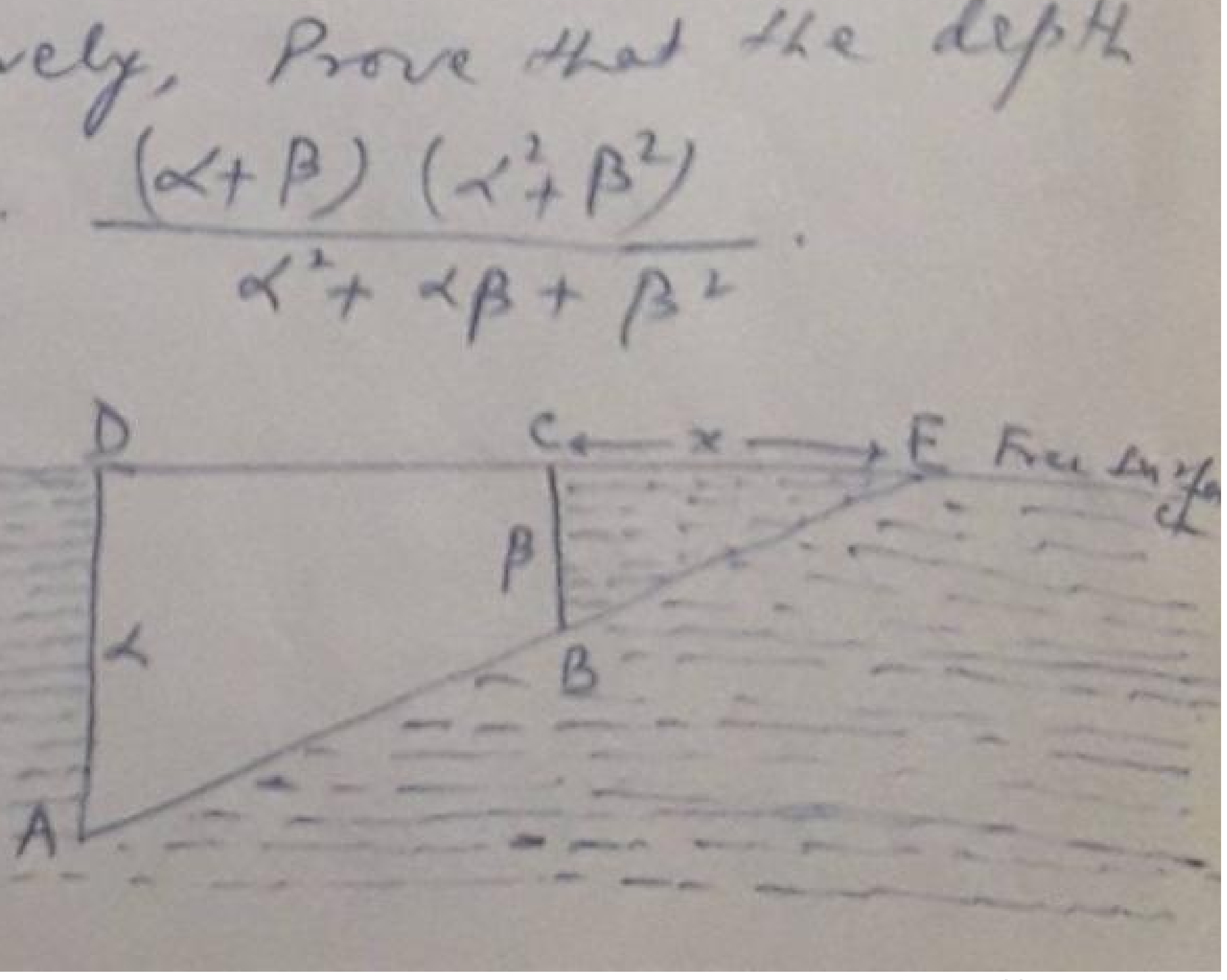
$$\bar{Z} = \frac{T_1Z_1 + T_2Z_2}{T_1+T_2}, \text{ formula}$$

$$= \frac{\frac{1}{6}\rho g h^2 a\cdot\frac{h}{2} + \frac{1}{3}\rho g h^2 b\cdot\frac{3}{4}h}{\frac{1}{6}\rho g h^2 a + \frac{1}{3}\rho g h^2 b} = \frac{\frac{1}{12}\rho g h^3(a+3b)}{\frac{1}{6}\rho g h^2(a+2b)}$$

$$= \frac{a+3b}{a+2b}\cdot\frac{h}{2} \quad \text{pr}$$

Ex-2 A lamina in the shape of a quadrilateral ABCD has the side CD in the surface, and the sides AD, BC vertical and of lengths α and β respectively, Prove that the depth of its centre of pressure is $\dfrac{1}{2}\cdot\dfrac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}$.

Sol^n: Let the side AB be produced to meet the free surface in E, s.t. $CE = x$ (say)



Since the triangles ADE and BCE are similar, therefore $\dfrac{AD}{BC} = \dfrac{DE}{CE}$

$\Rightarrow \dfrac{\alpha}{\beta} = \dfrac{DE}{x} \quad \therefore DE = \dfrac{\alpha}{\beta}x$

Let ρ be the density of the liquid.

Now the thrust on the $\Delta ADE = \rho g \cdot \frac{\alpha}{3} \cdot \frac{1}{2} \cdot DE \cdot AD$

$$= \rho g \cdot \frac{\alpha}{3} \cdot \frac{1}{2} \cdot \frac{\alpha}{\beta} \cdot x \cdot \alpha = T_1 \text{ (say)}$$

and the depth of the C.P of the ΔADE

$$= \frac{\alpha}{2} = Z_1 \text{ (say)}.$$

Also the thrust on the $\Delta BCE = \rho g \cdot \frac{\beta}{3} \cdot \frac{1}{2} \cdot CE \cdot BC$

$$= \rho g \cdot \frac{\beta}{3} \cdot \frac{1}{2} x \beta = T_2 \text{ (say)}.$$

and the depth of the C.P. of the $\Delta BCE = \frac{\beta}{2} = Z_2$ (say).

Quadrilateral $ABCD = \Delta AED - \Delta BEC$.

If $\bar{Z}$ be the required depth of the C.P of the quadrilateral $ABCD$,

$$\bar{Z} = \frac{T_1 Z_1 - T_2 Z_2}{T_1 - T_2}, \text{ formula.}$$

$$= \frac{\frac{1}{6}\rho g \frac{\alpha^3 x}{\beta} \cdot \frac{\alpha}{2} - \frac{1}{6}\rho g \beta^2 x \cdot \frac{\beta}{2}}{\frac{1}{6}\rho g \frac{\alpha^3 x}{\beta} - \frac{1}{6}\rho g \beta^2 x}$$

$$= \frac{\frac{1}{12}\rho g x \left(\frac{\alpha^4}{\beta} - \beta^3\right)}{\frac{1}{6}\rho g x \left(\frac{\alpha^3}{\beta} - \beta^2\right)} = \frac{1}{2} \cdot \frac{\alpha^4 - \beta^4}{\alpha^3 - \beta^3}$$

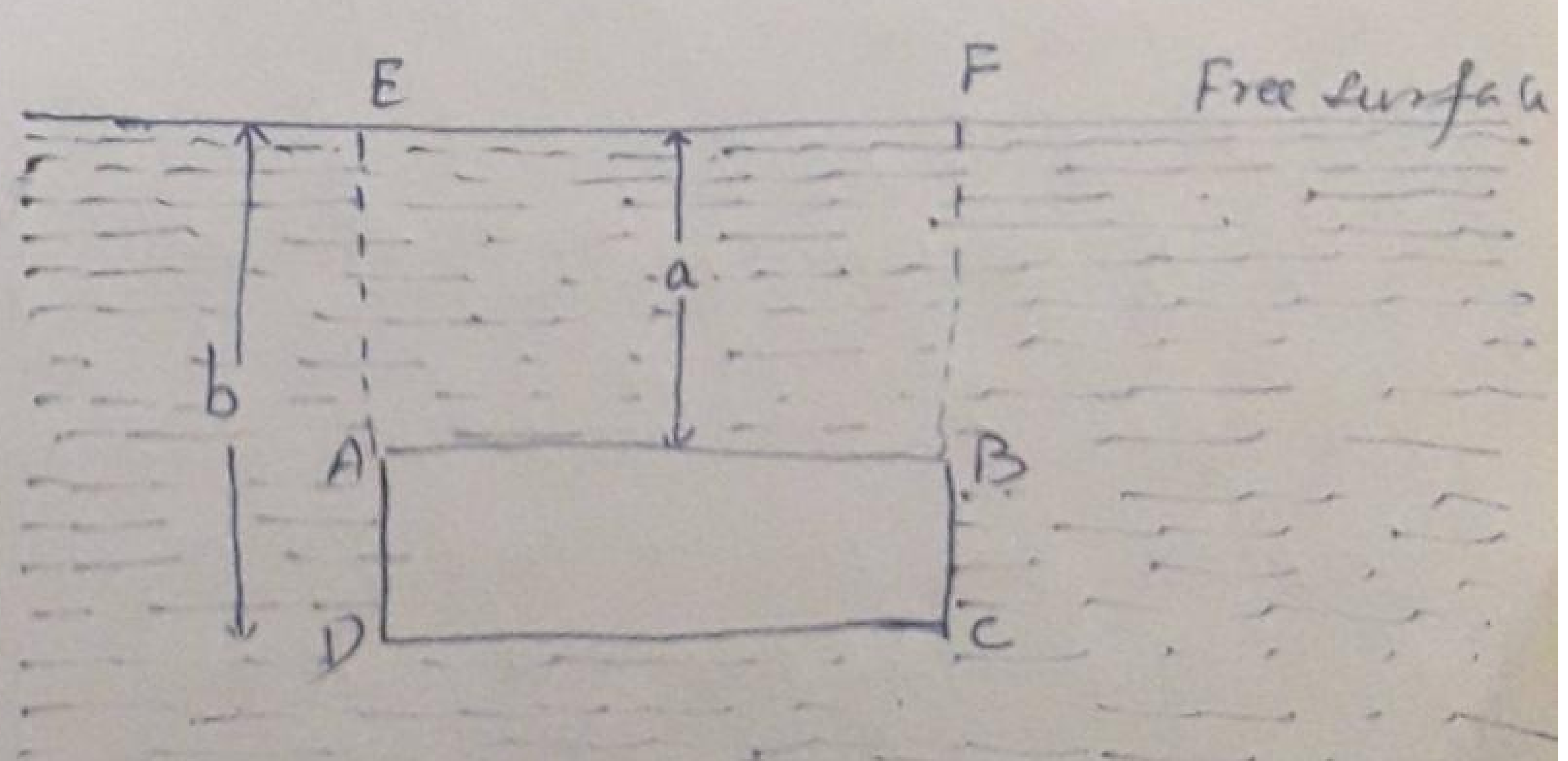
$$= \frac{1}{2} \frac{(\alpha-\beta)(\alpha+\beta)(\alpha^2+\beta^2)}{(\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)} = \frac{1}{2} \frac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}$$

Ex-3 Show that the depth below the surface of a liquid of the centre of pressure of a rectangle two of whose sides are horizontal and at depths a and b is

$$\frac{2}{3} \cdot \frac{a^2+ab+b^2}{a+b}$$

Soln:- Produce DA and CB to meet the free surface in E and F respectively

Let $AB = h = CD$, and ρ = the density of the liquid.



Now the rectangle $ABCD$ = rectangle $EFCD$ − rectangle $EFBA$.

The thrust on the rectangle EFCD $= \rho g \cdot \frac{b}{2} \cdot hb = T_1$ (say).
and the depth of the C.P. $= \frac{2}{3} b = z_1$ (say).
The thrust on the rectangle EFBA $= \rho g \cdot \frac{a}{2} \cdot ha = T_2$ (say).
and the depth of the C.P $= \frac{2}{3} a = z_2$ (say).

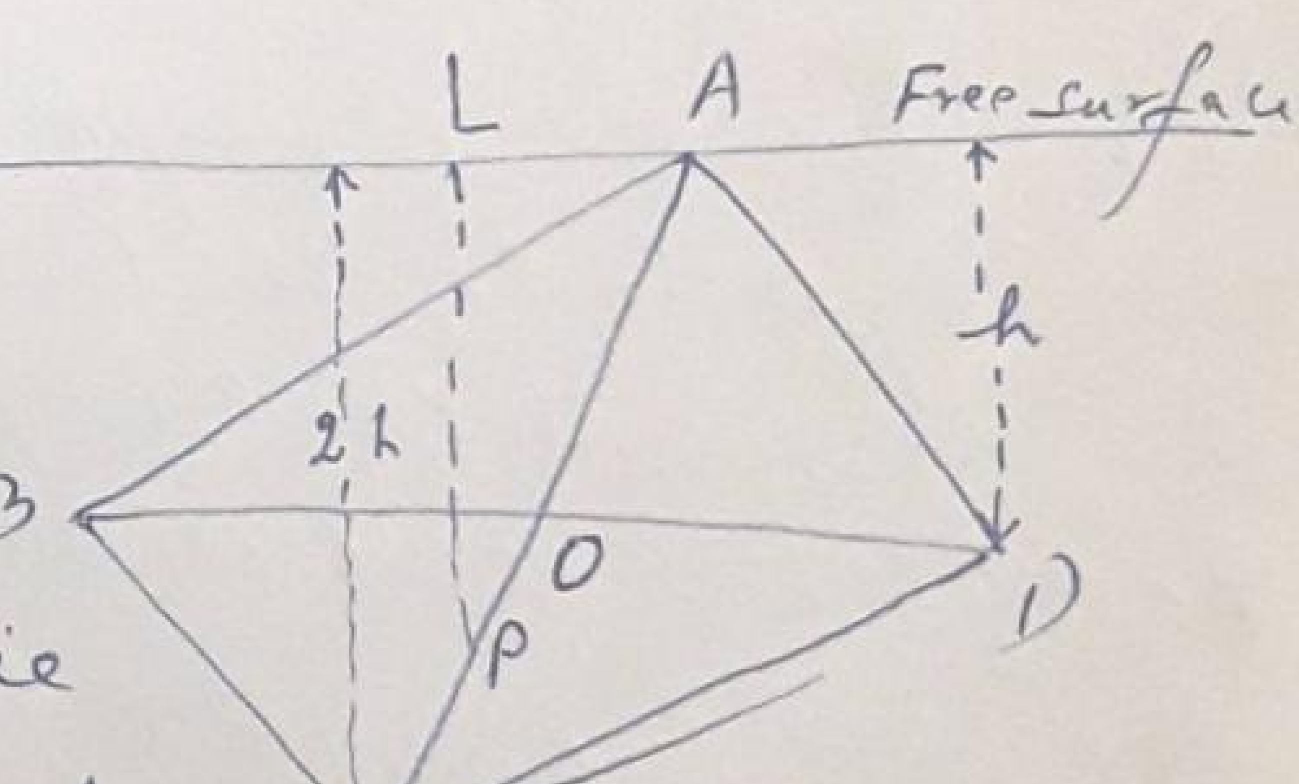
If $\bar{z}$ be the depth of the centre of pressure of the rectangle ABCD, $\bar{z} = \dfrac{T_1 z_1 - T_2 z_2}{T_1 - T_2}$, (formula)

i.e. $$\bar{z} = \frac{\frac{1}{2} \rho g b^2 h \cdot \frac{2}{3} b - \frac{1}{2} \rho g a^2 h \cdot \frac{2}{3} a}{\frac{1}{2} \rho g b^2 h - \frac{1}{2} \rho g a^2 h}$$

or $$\bar{z} = \frac{2}{3} \cdot \frac{\rho g h (b^3 - a^3)}{\rho g h (b^2 - a^2)} = \frac{2}{3} \frac{\cancel{(b-a)} (b^2 + ba + a^2)}{\cancel{(b-a)} (b+a)}$$

Hence $$\bar{z} = \frac{2}{3} \cdot \frac{b^2 + ba + a^2}{b + a}$$

Ex-4 A parallelogram ABCD is immersed in a liquid with A in the surface and BD horizontal. Prove that the centre of pressure (C.P) P lies on A such that AP : AC = 7 : 12.



Sol^n:- Here the area of the parallelogram = area of ΔABD + area of ΔBCD.
Since the diagonals of a parallelogram bisect each other, therefore the centre of pressure of Δ^s ABD and BCD will lie on the medians AO and CO respectively.
Hence the C.P of the parallelogram will also lie on AC.

$\because \Delta ABD \equiv \Delta BCD$,

$\therefore$ area of ΔABD = area of $\Delta BCD = S$ (say).

Let ρ be the density of the liquid, and h be the depth of BD below the free surface.
Then $2h$ is the depth of C below the free surface.
Now the thrust on $\Delta ABD = \rho g \cdot \frac{2}{3} h \cdot S = T_1$ (say)
and the depth of the C.P of $\Delta ABD = \frac{3}{4} h = z_1$ (say).
Also the thrust on $\Delta BCD = \rho g \left(\frac{h + h + 2h}{3}\right) S = \frac{4}{3} \rho g h \cdot S = T_2$ (say)

and the depth of the C.P of $\Delta BCD = \dfrac{\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha}{2(\alpha+\beta+\gamma)}$ (formula)

$$= \frac{h^2+h^2+(2h)^2+h\cdot h+h\cdot 2h+2h\cdot h}{2(h+h+2h)} = \frac{11}{8}h = Z_2 \text{ (say)}.$$

Hence the depth of P, the C.P. of the parallelogram ABCD below the surface $= \dfrac{T_1 Z_1 + T_2 Z_2}{T_1+T_2}$ (Formula)

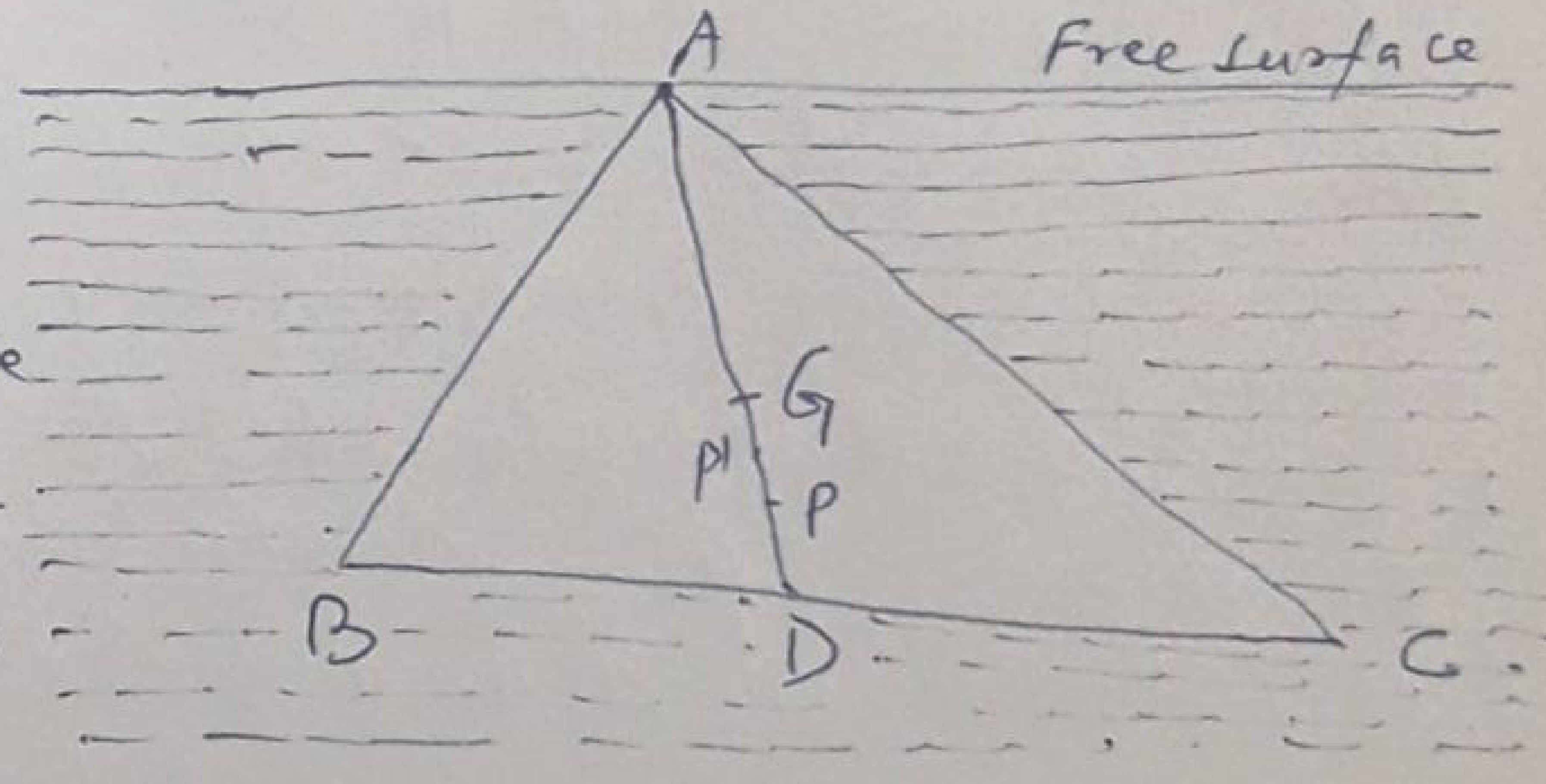
i.e. $PL = \dfrac{\frac{2}{3}\rho g h\cdot s\cdot\frac{3}{4}h + \frac{4}{3}\rho g h\cdot s\cdot\frac{11}{8}h}{\frac{2}{3}\rho g h\cdot s + \frac{4}{3}\rho g h\cdot s} = \dfrac{\rho g h^2\cdot s\left(\frac{1}{2}+\frac{11}{6}\right)}{\rho g h\cdot s\left(\frac{2}{3}+\frac{4}{3}\right)} = \dfrac{7}{6}h$

Also $CM = 2h$.

$\because$ Δ^s APL and ACM are similar,

$\therefore \dfrac{AP}{AC} = \dfrac{PL}{CM} = \dfrac{7/6\,h}{2h} = \dfrac{7}{12}$

Ex 5 A triangle of height h is immersed in a liquid with the base horizontal and vertex in the surface. If the atmospheric pressure is equivalent to a head of H feet of the liquid, prove that the centre of pressure is raised to a height $\dfrac{hH}{4(2h+3H)}$ in the plane of the triangle.



Sol:- If the atmospheric pressure be neglected, then the depths of the C.G and the C.P of the triangle ABC below the free surface are $\frac{2}{3}h$ and $\frac{3}{4}h$ respectively, where h is the depth of BC.

If the atmospheric pressure is taken into consideration, then the thrusts acting on the triangle are:

<i> the thrust $\rho g\cdot\frac{2}{3}h\cdot\Delta$ acting at P, where Δ = area of ΔABC, and depth of P below $A = \frac{3}{4}h$, and

an additional thrust $\rho g H\cdot\Delta$ acting at G, where H is the height of the same liquid superimposed on the surface of the given liquid, and depth of G below $A = \frac{2}{3}h$.

If $\bar{Z}$ be the depth of C.P (P') in the new position, $\bar{Z} = \dfrac{\rho g\cdot\frac{2}{3}h\cdot\Delta\frac{3}{4}h + \rho g H\cdot\frac{2}{3}h}{\rho g\cdot\frac{2}{3}h\Delta + \rho g H\cdot\Delta}$

$= \dfrac{\rho g h\Delta\left(\frac{h}{2}+\frac{2H}{3}\right)}{\rho g\Delta\left(\frac{2h}{3}+H\right)}$

$= \dfrac{h}{2}\cdot\dfrac{3h+4H}{2h+3H}$

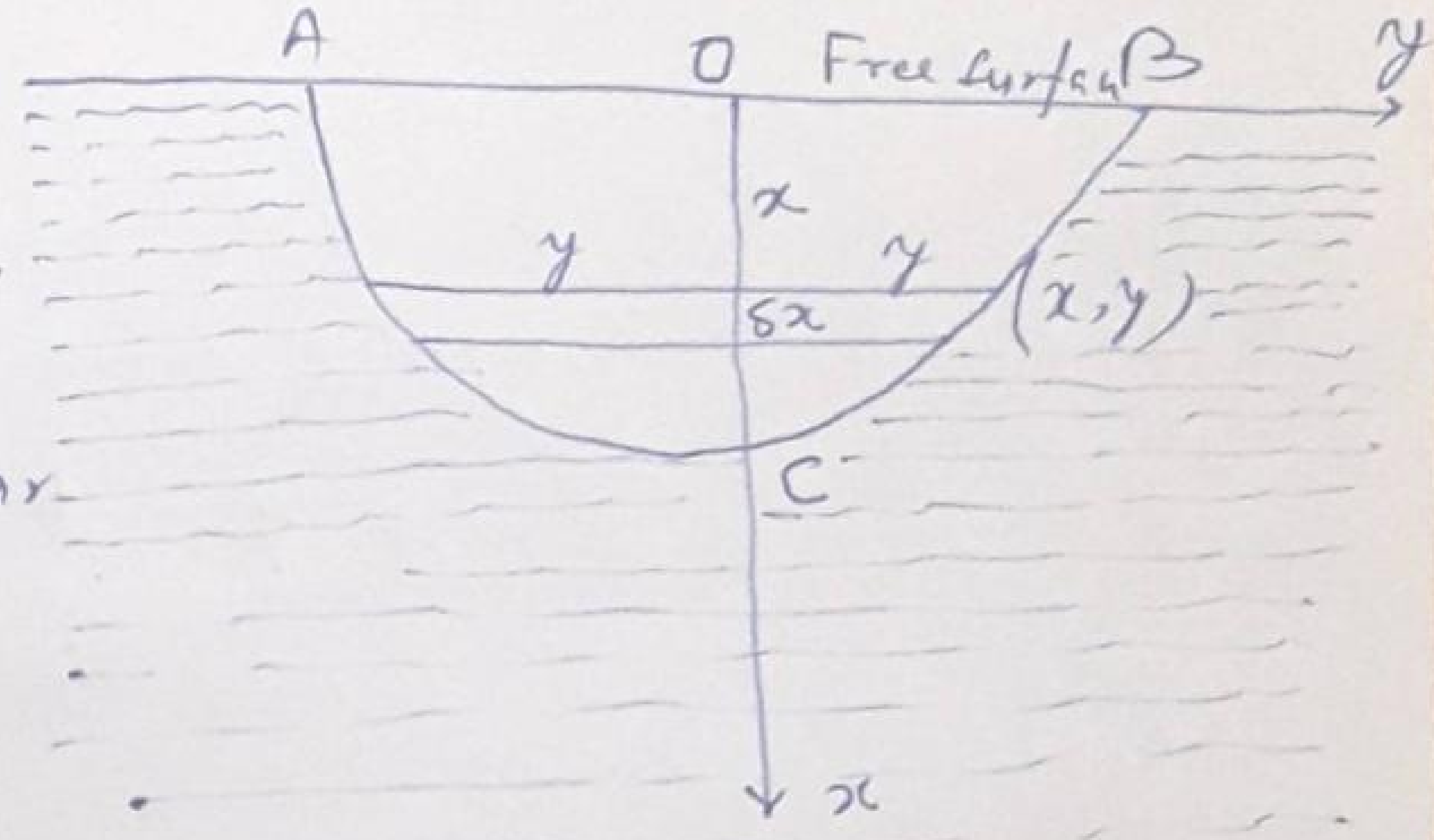
Hence the height through which the C.P. is raised $= \dfrac{3}{4}h - \dfrac{h}{2}\cdot\dfrac{3h+4H}{2h+3H}$

= depth of P − depth of P'

$= \dfrac{hH}{4(2h+3H)}$

Ex-6 A semi-circular lamina is immersed in a liquid with the diameter in the surface. Find the centre of pressure.

Soln:- Let OC be a radius of the circle perpendicular to AOB.



Take OC and OB and x and y axes respectively.

Then the equation of the circular arc is $x^2+y^2=a^2$ —— (1)

where a is the radius of the circle.

The lamina is symmetrical about OC, therefore, by symmetry, the C.P. of the lamina will lie on OC. $\therefore \bar{y}=0$

The pressure on the elementary strip is at the middle point whose co-ordinates are $\left(x, \frac{y}{2}\right)$.

$$\therefore \bar{y} = \frac{\int_0^a \frac{y}{2}\cdot \rho g x\cdot \sqrt{a^2-x^2}\,dx}{\int_0^a \rho g x\cdot \sqrt{a^2-x^2}\,dx} = \frac{1}{2}\frac{\int_0^a \sqrt{a^2-x^2}\cdot x\cdot \sqrt{a^2-x^2}\,dx}{\int_0^a x\sqrt{a^2-x^2}\,dx}$$

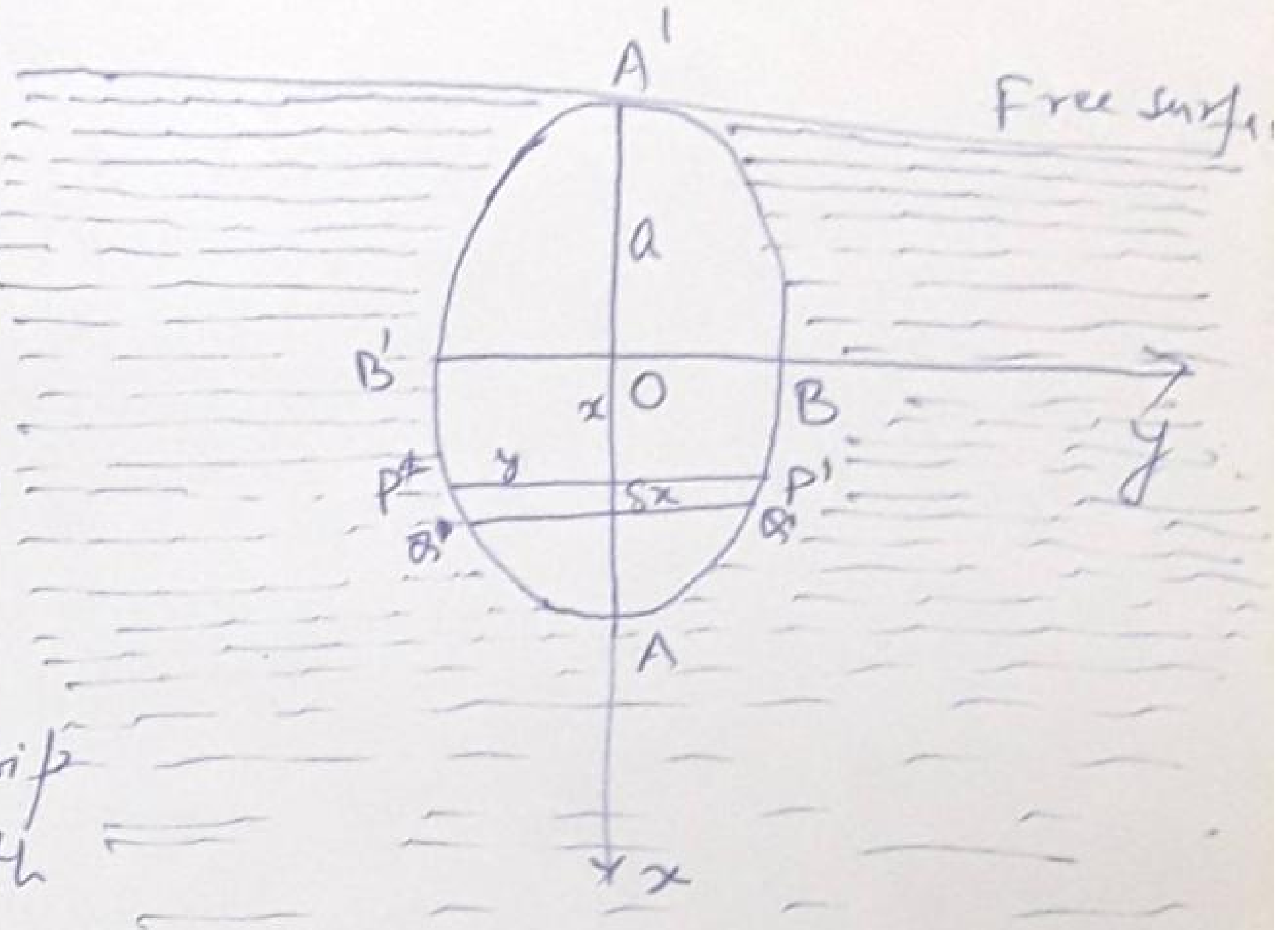
$$= \frac{1}{2}\cdot \frac{\int_0^{\pi/2} a\sin\theta\,(a^2-a^2\sin^2\theta)\cdot a\cos\theta\,d\theta}{\int_0^{\pi/2} a\sin\theta\,\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta}$$

$$= \frac{a}{2}\cdot \frac{\int_0^{\pi/2}\sin\theta\cos^3\theta\,d\theta}{\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta} = \frac{a}{2}\cdot\frac{\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2}}{\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}} = \frac{3}{8}a$$

Hence the co-ordinates of C.P of the quadrant are $\left(\frac{3\pi a}{16}, \frac{3a}{a}\right)$.

Ex-7 An ellipse is just immersed with its major axis vertical. Show that if C.P coincides with the lower focus, the eccentricity of the ellipse is $1/4$.

Sol^n:- Taking the major and minor axes of the ellipse as the axes of co-ordinates, the eq^n of the ellipse is



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2} \quad \text{———} (1)$$

where $AA'=2a$ and $BB'=2b$.

Now consider an elementary strip $PP'Q'Q^*$ of thickness δx at a depth x below O, the centre of the ellipse.

Then δS = area of the strip

$$= 2y\,\delta x = \frac{2b}{a}\sqrt{a^2-x^2}\,\delta x. \text{ [from (1)]}$$

and p = pressure per unit area $= \rho g(a+x)$

Clearly the C.P of the ellipse lies on the major axis, $\therefore \bar{y}=0$.

If $\bar{x}$ be the depth of C.P. of the ellipse from O, then

$$\bar{x}=\frac{\int x\,p\,ds}{\int p\,ds}=\frac{\int_{-a}^{a} x\,\rho g(a+x)\frac{2b}{a}\sqrt{a^2-x^2}\,dx}{\int_{-a}^{a}\rho g(a+x)\frac{2b}{a}\sqrt{a^2-x^2}\,dx}$$

$$=\frac{a\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx+\int_{-a}^{a}x^2\sqrt{a^2-x^2}\,dx}{a\int_{-a}^{a}\sqrt{a^2-x^2}\,dx+\int_{-a}^{a}x\sqrt{a^2-x^2}\,dx}$$

If $f(x)$ is an odd function of x, then $\int_{-a}^{a}f(x)\,dx=0 \Rightarrow \int_{-a}^{a}x\sqrt{a^2-x^2}\,dx=0$

$$\therefore \bar{x}=\frac{\int_{-a}^{a}x^2\sqrt{a^2-x^2}\,dx}{a\int_{-a}^{a}\sqrt{a^2-x^2}\,dx}=\frac{a\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta}{\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta}$$

$\left\{\begin{array}{l}\text{Put } x=a\sin\theta \Rightarrow dx=a\cos\theta\,d\theta\\ \text{when } x=a,\ \theta=\pi/2\\ x=-a,\ \theta=-\pi/2\end{array}\right.$

$$=\frac{a}{4}\cdot\frac{\int_{-\pi/2}^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta}{\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta}=\frac{a}{4}\cdot\frac{\pi}{\pi}=\frac{a}{4}$$

But the C.P. coincides with the focus. $\therefore \bar{x}=ae$ i.e. $\frac{a}{4}=ae$

Hence the eccentricity of the ellipse $=1/4$