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# Degree - 3 ( H )

## CHAPTER:- DIFFERENTIAL EQUATIONS

### Topic:- Non- linear Differential Equations of order one

# Theory + Problems

# STANDARD Form:

1.

2.

3.

4.

BY Professor DR. MOHAMMAD EQBALU ZAFAR

General method of solving equations of order one but of any degree.

Charpit's method.

Let the given equation be $f(x, y, z, p, q) = 0$ — (1)

We know that $dz = p\,dx + q\,dy$ — (2)

Let $F(x, y, z, p, q) = 0$ — (3)

such that when the values of $p$ and $q$ obtained by solving (1) and (3) are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), diff (1) and (3) w.r.to $x$ and $y$. Then we get.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \frac{\partial q}{\partial x} = 0,$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial y} = 0,$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial F}{\partial q} \frac{\partial q}{\partial y} = 0.$$

Eliminating $\frac{\partial p}{\partial x}$ from the first pair of these equation, we get.

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x}\right) \frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial q} \frac{\partial q}{\partial x}\right) \frac{\partial f}{\partial p} = 0$$

or
$$\left(\frac{\partial f}{\partial x} \frac{\partial F}{\partial p} - \frac{\partial F}{\partial x} \frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z} \frac{\partial F}{\partial p} - \frac{\partial F}{\partial z} \frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial q} \cdot \frac{\partial f}{\partial p}\right) \frac{\partial q}{\partial x} = 0 \quad \text{— (4)}$$

* Since $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$,

the last term in (4) is the same as that in (5), except for a minus sign and hence they cancel on adding (4) and (5)

Similarly, the elimination of $\frac{\partial q}{\partial y}$ between the second pair gives

$$\left(\frac{\partial f}{\partial y}\cdot\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \quad \text{--- (5)}$$

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* Therefore (4) + (5) and re-arranging

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0 \quad \text{--- (6)}$$

This is a linear equation of the first order to obtain the desired function F. integral of (6) is obtained by solving the auxiliary eqns.

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}}$$

$$= \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad \text{--- (7)}$$

Since any of the integrals of (7) will satisfy (6), an integral of (7) which involves p or q (both) will serve along with the given eqn to find p & q.

Solve by applying Charpit's method.

Ex-1 $(p^2+q^2)y = qz$

sol^n: Given equation $f(x,y,z,p,q)=0$ is —

$$f \equiv (p^2+q^2)y - qz = 0 \quad \text{— (1)}$$

Now charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad \text{— (2)}$$

But $\frac{\partial f}{\partial p} = 2py$ , $\frac{\partial f}{\partial x} = 0$ , $\frac{\partial f}{\partial z} = -q$.

$\frac{\partial f}{\partial q} = 2qy - z$ , $\frac{\partial f}{\partial y} = p^2+q^2$

Putting these values in (2) and simplifying, we get.

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

From I^st & 2^nd relation $\frac{dp}{-\cancel{p}q} = \frac{dq}{p^{\cancel{2}}} \Rightarrow p\,dp + q\,dq = 0$.

Integrating, $p^2+q^2 = a^2$, say. — (3)

Using (3), (1) gives $a^2 y = qz \Rightarrow q = a^2y/z$

Putting this value of $q$ in (3), we get.

$$p = \sqrt{a^2-q^2} = \sqrt{a^2 - \frac{a^4y^2}{z^2}} = \frac{a}{z}\sqrt{z^2-a^2y^2}$$

Now putting these values of $p$ and $q$ in $dz = p\,dx + q\,dy$,

$$dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

integrating, $\sqrt{z^2-a^2y^2} = ax+b$

i.e. $z^2 - a^2y^2 = (ax+b)^2$ — (4) is the required complete integral.

Singular Integral: Diff (4) partially w.r.to a and b. we have $0 = 2ay^2 + 2(ax+b)x$ — (5) (25)

$0 = 2(ax+b)$ — (6)

Eliminating a and b between (4), (5) and (6), we get $z=0$ which clearly satisfies (1) and hence it is the Singular integral.

General Integral: Replacing b by $\phi(a)$ in (4), we get.

$$z^2 - a^2y^2 = (ax + \phi(a))^2 \quad \text{— (7)}$$

Diff (7) partially w.r.to a, we get.

$$-2ay^2 = 2[ax + \phi(a)] \cdot [x + \phi'(a)] \quad \text{— (8)}$$

G.I is obtained by eliminating a from (7) and (8)

Ex-2 Find C.I of $p(1+q^2) + (b-z)q = 0$

Soln: Here $f \equiv p(1+q^2) + (b-z)q = 0$ — (1)

Charpit's auxiliary eqns are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad \text{— (2)}$$

But ~~$\frac{\partial f}{\partial p} = q$~~ $\frac{dp}{pq} = \frac{dq}{q^2} = \frac{dz}{3pq^2 + p + (b-z)q} = \frac{dx}{q^2+1} = \frac{dy}{-z+b+2pq}$

From first two fractions $\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p + \log c = \log q \Rightarrow q = pc$

Put this value in (1)

$$p(1+p^2c^2) + (b-z)pc = 0$$

$$p = \sqrt{[c(z-b)-1]}/c$$

$\therefore q = pc$ gives $q = \sqrt{c(z-b)-1}$

Putting these values of p and q in $dz = p\,dx + q\,dy$.

$$dz = [c(z-b)-1]\left(\frac{dx}{c} + dy\right)$$

or $\frac{c\,dz}{\sqrt{c(z-b)-1}} = dx + c\,dy$ Integ: $2\sqrt{[c(z-b)-1]} = x + cy + a$

is the complete integral, a and c are arbitrary const.

**Ex-3** Find C.I of $2xz - px^2 - 2qxy + pq = 0$

Soln: Here $f \equiv 2xz - px^2 - 2qxy + pq = 0$ — (1)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad \text{— (2)}$$

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 + 2xyq - 2pq}$$

The second fraction gives $dq = 0$ so that $q = a$.

Putting $q = a$ in (1), we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

Putting these values in $dz = p\,dx + q\,dy$.

$$dz = \frac{2x(z-ay)}{x^2-a}\,dx + a\,dy \quad \text{or} \quad \frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}$$

Integrating. $\log(z - ay) = \log(x^2 - a) + \log b$.

$\therefore (z - ay) = b(x^2 - a)$ is the C.I where $a$ and $b$ are const.

**Ex-4** Find the complete integral of the following —

(i) $q = px + p^2$ (ii) $q = -px + p^2$

Soln— (i) Here $f \equiv q - px - p^2 = 0$ — (1)

Charpit's auxiliary equations are —

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\therefore \frac{dp}{-p} = \frac{dq}{0} = \frac{dz}{-p(-x-2p) - q} = \frac{dx}{-(-x-2p)} = \frac{dy}{-1}.$$

The Second fraction gives $dq = 0$ so that $q = a$.

Putting $q = a$ in (1) $p^2 + px - a = 0 \Rightarrow p = \frac{1}{2}\left[-x \pm \sqrt{x^2 + 4a}\right]$

Putting the value of $p$ and $q$ in $dz = p\,dx + q\,dy$.

$$dz = \frac{1}{2}\left[-x \pm \sqrt{x^2 + 4a}\right]dx + a\,dy$$

Integrating, the required complete integral is

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + 2a\log\left\{x+\sqrt{x^2+4a}\right\}\right] + ay + b$$

(ii) $f \equiv q + px - p^2 = 0$ Proceed as above

complete integral is

$$x = \frac{-x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + 2a\log\{x+\sqrt{x^2+4a}\}\right] + ay + b.$$

Ex-5 Find the complete integrals of —

(i) $px + qy + pq = 0$ (ii) $px + qy = pq$

Soln: Here $f \equiv px + qy + pq = 0$ — (1)

Charpit's auxiliary equations are —

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad — (2)$$

$$\therefore \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x+q) - q(y+p)} = \frac{dx}{-(x+q)} = \frac{dy}{-(y+p)}.$$

The first two fractions give $\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log a + \log q$ $\therefore p = aq.$

Put $p = aq$ in (1) $aqx + qy + aq^2 = 0 \Rightarrow ax + y + aq = 0$

$$q = -\frac{(y+ax)}{a} \text{ and hence } p = -(y+ax)$$

Putting the value of $p$ and $q$ in $dz = p\,dx + q\,dy$

$$dz = -(y+ax)\,dx - \left(\frac{y+ax}{a}\right)dy$$

or $a\,dz = -(y+ax)(dy + a\,dx)$

Integrating $az = -\frac{1}{2}(y+ax)^2 + b$ is the required C.I.

(ii) $az = -\frac{1}{2}(y+ax)^2 + b.$

Ex-6 Find C.I. $pxy + pq + qy = yz$.

Sol$^n$: Here $f \equiv pxy + pq + qy - yz = 0$ — (1)

$\therefore$ The charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad -(2)$$

$$\therefore \frac{dp}{0} = \frac{dq}{(px+q)+qy} = \frac{q\,dz}{-p(xy+q) - q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}$$

The first fraction gives $dp = 0$ so that $p = a$.

Put $p = a$ in (1) $axy + aq + qy = yz$

or $q(a+y) = y(z - ax)$ $\therefore q = \frac{z - ax}{a+y}$

Putting the value of $p$ and $q$ in $dz = p\,dx + q\,dy$.

$$dz = a\,dx + \frac{y(z-ax)}{a+y}dy$$

or $$\frac{dz - a\,dx}{z - ax} = \frac{y\,dy}{a+y}$$

or $$\frac{dz - a\,dx}{z - ax} = \left(1 - \frac{a}{a+y}\right)dy$$

Integrating $\log(z - ax) = y - \log(a+y) + \log b$

or $\log(z - ax) + \log(a+y) = y + \log b$

$\log(z-ax)(a+y) = \log(b\cdot e^y) \Rightarrow (z-ax)(a+y) = b\cdot e^y$ is the C.I.

Ex-7 Find C.I of $p^2 + q^2 - 2px - 2qy + 1 = 0$

Sol$^n$: Here $f \equiv p^2 + q^2 - 2px - 2qy + 1 = 0$.

charpit's $$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x) - q(2q-2y)} = \frac{dx}{-(2p-2x)} = \frac{dy}{-(2q-2y)}$$

The first-two fractions give:

$\frac{dp}{p} = \frac{dq}{q}$ or $\log p = \log q + \log a$

$\therefore p = aq$

Putting $p = aq$ in ① $a^2q^2 + q^2 - 2aqx + 2qy + 1 = 0$

or $(a^2+1)q^2 - 2(ax+y)q + 1 = 0$

$$q = \frac{2(ax+y) \pm \sqrt{4(ax+y)^2 - 4(a^2+1)}}{2(a^2+1)}$$

Again, $$p = aq = a\,\frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}$$

Putting the value of $p$ and $q$ in $dz = p\,dx + q\,dy$

or $$dz = \frac{(ax+y) \pm \sqrt{\{(ax+y)^2 - (a^2+1)\}}}{a^2+1}\,(a\,dx + dy)$$

Put $ax + y = v$ so that $a\,dx + dy = dv$.

Then ② gives. $(a^2+1)\,dz = \left[v \pm \sqrt{\{v^2 - (a^2+1)\}}\right] dv$

$$\therefore (a^2+1)z = \frac{1}{2}v^2 \pm \frac{1}{2}v\sqrt{\{v^2 - (a^2+1)\}}$$
$$\pm \frac{1}{2}(a^2+1)\log\left[v + \sqrt{\{v^2 - (a^2+1)\}}\right] + b$$

is the complete integral. where $v = ax + b$.

Ex-8 Solve $z = px + qy + p^2 + q^2$

Sol$^n$: Here $f \equiv z - px - qy - p^2 - q^2 = 0$ —— ①

Charpit's auxiliary equations are –

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p(x-2p) - q(-y-2q)} = \frac{dx}{-(-x-2p)} = \frac{dy}{-y-2q}$$

The first fractions gives $p = a$

Similarly second " " $q = b$

Putting in ① $z = ax + by + a^2 + b^2$ is the C.I.

Ex-9 Find C.I. of $p^2+q^2-2px-2qy+2xy=0$.

Sol$^n$: Here $f \equiv p^2+q^2-2px-2qy+2xy=0$ — (1)

Charpit's auxiliary eq$^{ns}$ are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}} \quad (2)$$

$$\Rightarrow \frac{dp}{-2p+2y}=\frac{dq}{-2q+2x}=\frac{dx}{2x-2p}=\frac{dy}{2y-2q}$$

which give: $\dfrac{dp+dq}{2(x+y-p-q)}=\dfrac{dx+dy}{2(x+y-p-q)}$

or $dp+dq=dx+dy$ i.e. $dp-dx+dq-dy=0$

Integrating $(p-x)+(q-y)=a$ — (3)

Re-writing (1), $(p-x)^2+(q-y)^2=(x-y)^2$ — (4)

putting the value of $(q-y)$ from (3) in (4)

$$(p-x)^2+[a-(p-x)]^2=(x-y)^2$$

or $2(p-x)^2-2a(p-x)+\{a^2-(x-y)^2\}=0$

$$\therefore p-x=\frac{2a\mp\sqrt{[4a^2-4\cdot2\cdot\{a^2-(x-y)^2\}]}}{4}$$

$$p=x+\frac{1}{2}\left[a\pm\sqrt{\{2(x-y)^2-a^2\}}\right]$$

then (3) gives $q=y+\frac{1}{2}\left[a\mp\sqrt{\{2(x-y)^2-a^2\}}\right]$

Putting these values of $p$ and $q$ in $dz=p\,dx+q\,dy$, we get

$$dz=x\,dx+y\,dy+\frac{a}{2}(dx+dy)\pm\frac{1}{2}\sqrt{\{2(x-y)^2-a^2\}}\,(dx-dy)$$

or $dz=x\,dx+y\,dy+\frac{a}{2}(dx+dy)\pm\frac{1}{\sqrt{2}}\cdot\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}\,(dx-dy)$

Integrating, C.I is $z=\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{a}{2}(x+y)\pm\dfrac{1}{\sqrt{2}}\left(\dfrac{(x-y)}{2}\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}-\dfrac{a^2}{4}\log\left[(x-y)+\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}\right]\right)$

or $2z=x^2+y^2+ax+ay\pm\dfrac{1}{\sqrt{2}}\left((x-y)\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}-\dfrac{a^2}{2}\log\left[(x-y)+\sqrt{\left\{(x-y)^2-a^2/2\right\}}\right]\right)$

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H.W

1. $q = (z+px)^2$
2. $p = (z+qy)^2$
3. $z^2(p^2z^2+q^2) = 1$
4. $(p^2+q^2)y = qz$
5. $(p^2+q^2)x = pz$
6. $z = px+qy+pq$
7. $z = pq$
8. $yzp^2 = q$
9. $qz+p^2+qy+2y^2 = 0$
10. $2x(z^2q^2+1) = pz$ or $2x\left\{z^2\left(\frac{\partial z}{\partial x}\right)+1\right\} = z\frac{\partial z}{\partial x}$
11. $q = 3p^2$
12. $p-3x^2 = q^2-y$
13. $pq+x(2y+1)p+(y^2+y)q-(2y+1)z = 0$
14. $2(z+px+qy) = yp^2$
15. $z^2 = pqxy$
16. $px^5-4q^3x^2+6x^2z-2 = 0$
17. $xp+3yq = 2(z-x^2q^2)$
18. $(x^2-y^2)pq-xy(p^2-q^2)-1 = 0$
19. $2(pq+yp+qx)+x^2+y^2 = 0$
20. $p^2x+q^2y = z$