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Chapter:-
Sphere

Topic:- Solid geometry (3D)

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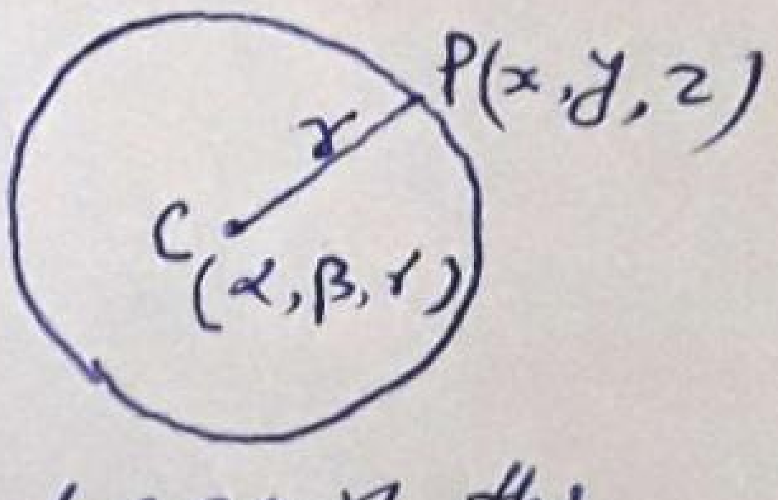
SPHERE

Defn: A sphere is the locus of a point which remains at a constant distant from a fixed point. The fixed point is called the centre and the constant distance the radius of the sphere.

(Art) Find the equation of the sphere whose centre is the point (α, β, γ) and radius equal to r.

Proof: Let $P(x, y, z)$ be any point on the sphere.



Join the centre C and P. Then CP = radius, r

Also $CP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$

$\therefore$ $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$ is the required eqn of the sphere.

In Particular, the eqn of the sphere whose centre is the origin and radius a. is $x^2 + y^2 + z^2 = a^2$.

(Art) The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere whose centre is $(-u, -v, -w)$ and radius $= \sqrt{u^2+v^2+w^2-d}$

Proof:

$$(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) = -d$$

$$(x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) = u^2 + v^2 + w^2 - d$$

or

$$(x+u)^2 + (y+v)^2 + (z+w)^2 = u^2 + v^2 + w^2 - d$$

which is of the form $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$

where $\left.\begin{array}{l}\alpha = -u \\ \beta = -v \\ \gamma = -w\end{array}\right\}$, radius, $r = \sqrt{u^2 + v^2 + w^2 - d}$

centre $= (-u, -v, -w)$

Thus the general equation of a sphere is s.t.

(i) it is the second degree in x, y, z

(ii) the co-efficient of x^2, y^2, z^2 are equal

and (iii) there are no terms containing yz, zx or xy.

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Art The equation of a sphere through four points.

Proof: Let the general equation of a sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

and let the co-ordinates of the four given points be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4).

Since the four points lie on the sphere (1), we have

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \quad \text{--- (2)}$$

$$x_2^2+y_2^2+z_2^2+2ux_2+2vy_2+2wz_2+d=0 \quad \text{--- (3)}$$

$$x_3^2+y_3^2+z_3^2+2ux_3+2vy_3+2wz_3+d=0 \quad \text{--- (4)}$$

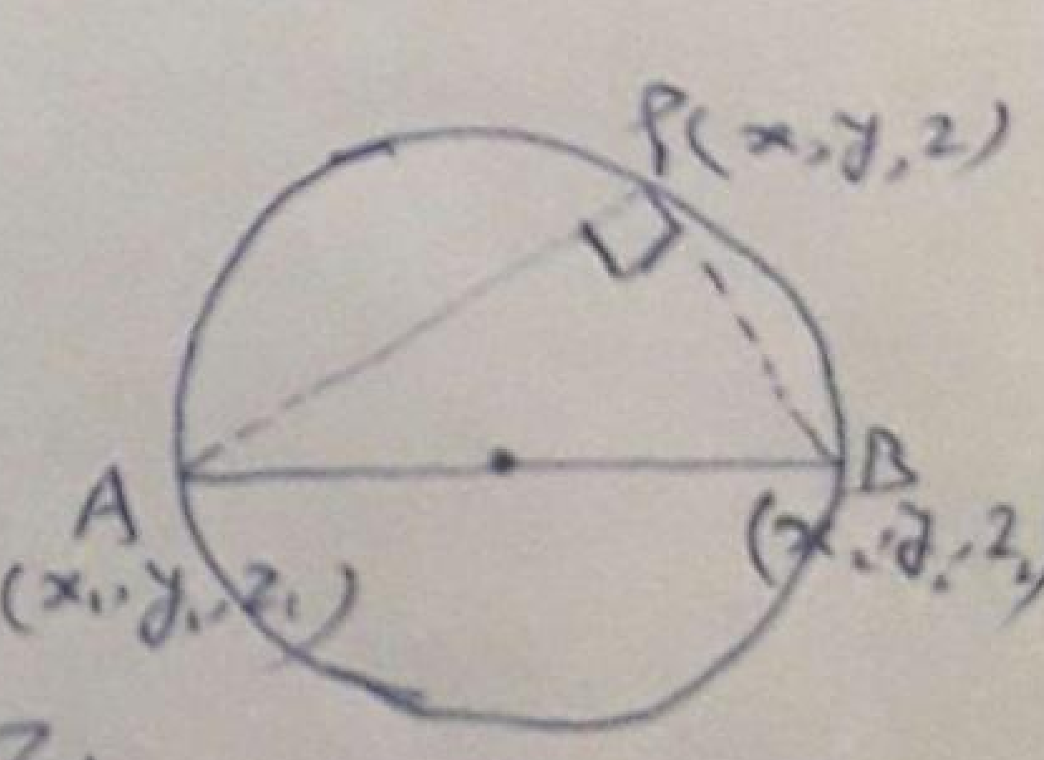
$$x_4^2+y_4^2+z_4^2+2ux_4+2vy_4+2wz_4+d=0 \quad \text{--- (5)}$$

Eliminating the four constants u, v, w, d from (1) to (5), we get

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

which is the required eqn of the sphere.

Art Find the eqn of the sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter.



Proof: Let $P(x,y,z)$ be any point on the sphere having $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ as ends of diameter. Then AP and BP are at right angles.

Now d.r.s of AP are $x-x_1,\ y-y_1,\ z-z_1$

and those of BP are $x-x_2,\ y-y_2,\ z-z_2$

Hence $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

is the required eqn.

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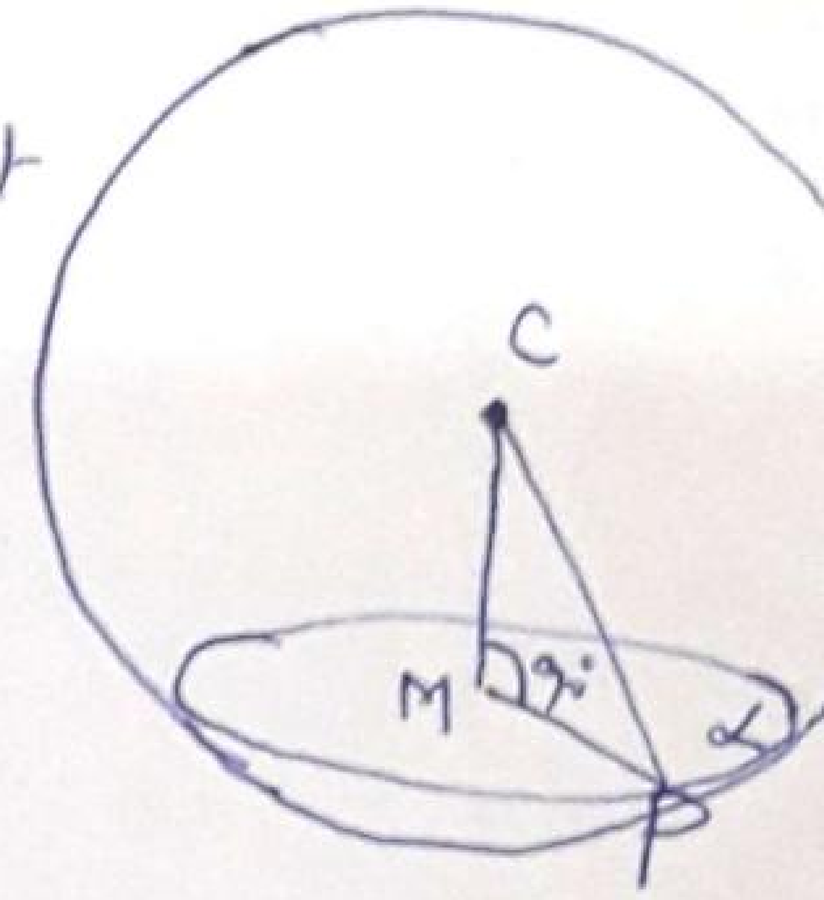
(Art) Prove that the section of a sphere by a plane is a circle.

Proof: Let α denote the section of the sphere by a plane and let P be any point common to the plane and the sphere.



Let C be the centre of the sphere.

From C draw CM perpendicular to the plane α. Join MP and CP.

Since CM is the perpendicular to the plane α, therefore it is perpendicular to every line lying in the plane

$\therefore$ $CM \perp MP$.

Now, from the $\triangle CMP$, $CP^2 = CM^2 + MP^2$

$\therefore$ $MP^2 = CP^2 - CM^2 =$ constant

Since CP is the radius of the sphere which is constant and CM is constant because C is fixed, being the centre of the sphere and M is fixed because it is the foot of the perpendicular from C on the plane α.

$\therefore$ MP = constant

Now M is fixed and P is a point in the plane such that MP is constant. Hence the locus of P is a circle whose centre is the point M and radius MP.

Corollary: (i) Centre and radius of the circle.

The foot of the perpendicular from the centre of the sphere to the plane is the centre of the circle. Thus

$(\text{radius of the circle})^2 = (\text{radius of the sphere})^2 - (\text{Perpendicular from the centre of the sphere on the plane})^2$

(ii) Great-Circle: The section of a sphere by a plane through the centre of the sphere is a <u>great circle</u>.

Its centre and radius are the same as that of the given sphere.

Curve of intersection of Two Spheres

(Art) Prove that the Centre of intersection of two spheres is a Circle.

Proof: Let $S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

and $S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

be the equations of two spheres. Now consider the equation $S_1-S_2=0$.

This is an eqn which is satisfied by the points common to both S_1 and S_2.

Also, $S_1-S_2 \equiv 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0$

which is of the first degree and therefore represents a plane.

Thus the points of intersection of two spheres are the same as those of the intersection of any one of the spheres and the plane $S_1-S_2\equiv 0$ and thus these points lie on a circle.

(Art) Sphere through a given Circle.

(i) The general eqn of a sphere passing through the common points of a given sphere and a plane.

Proof: Let $S\equiv x^2+y^2+z^2+2ux+2vy+2wz+K=0$

be the eqn of the sphere and let $P\equiv ax+by+cz+d=0$

be the eqn of the plane.

Consider the eqns $S+\lambda P=0$

For all values of λ, this eqn is satisfied by all those points which satisfy both the equations $S=0$ and $P=0$.

But $S=0$, $P=0$ taken together represents a circle. Hence the equation $S+\lambda P=0$ is satisfied by all those points which lie on the Circle $S=0$, $P=0$,

But the equation $S+\lambda P\equiv x^2+y^2+z^2+2ux+2vy+2wz+K+\lambda(ax+by+cz+d)=0$

is an eqn in which the Co-efficients of x^2, y^2, z^2 are equal and which does not Contain the terms xy, yz, zx

Hence the eqn $S+\lambda P=0$ represents a sphere through the circle whose eqns are $S=0$, $P=0$.

(iii) The general eqn of a sphere passing through the common points of two given spheres.

Proof: Let $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

be the equations of two given spheres.

Consider the eqn $S+\lambda S_1=0$

For different values of $\lambda\,(\neq -1)$ it represents a sphere, as there is no term containing xy, yz, zx in it and the co-efficients are of x^2, y^2, z^2 are equal.

Also it passes through the common points of $S=0$, $S_1=0$

Hence for different values of λ, $S+\lambda S_1=0$ represents a system of spheres passing through the circle whose eqns are $S=0, S_1=0$

— x —

Equation of the tangent plane

(Art 2) The equation of the tangent plane at any point (x_1,y_1,z_1) of the sphere $x^2+y^2+z^2=a^2$ is $xx_1+yy_1+zz_1=a^2$

Proof: If $P(x,y,z)$ be any point on the tangent plane at $P_1(x_1,y_1,z_1)$ to the given sphere, the direction ratios of P_1P are

$x-x_1,\ y-y_1,\ z-z_1$.

Also the d.rs of radius OP_1 are x_1-0, y_1-0, z_1-0.

Since OP_1 is normal to the tangent plane at P_1,

$OP_1 \perp P_1P$. $\therefore\ x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$

$xx_1+yy_1+zz_1 = x_1^2+y_1^2+z_1^2=a^2$ ($\because P_1(x_1,y_1,z_1)$ lies on the sphere)

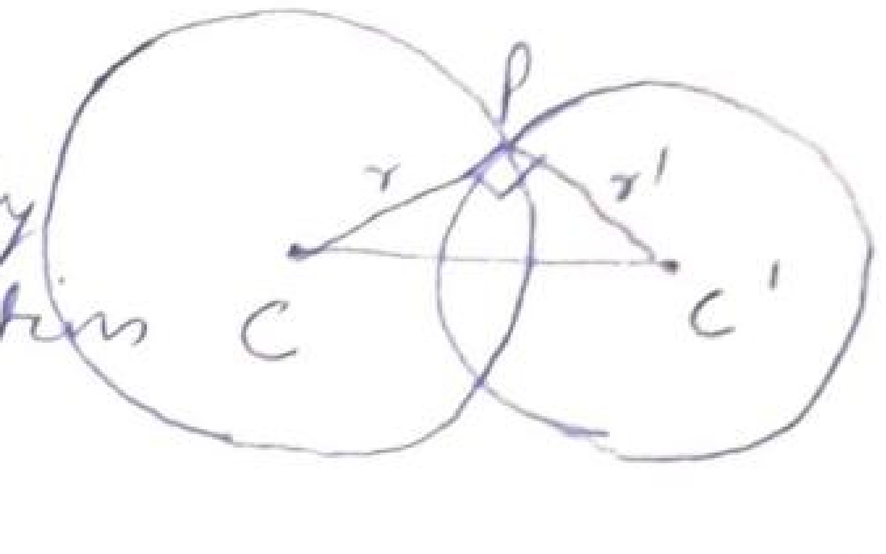
which is the desired eqn of the tangent plane.

Similarly, the tangent plane at (x_1,y_1,z_1) to the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

is $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

Orthogonal spheres:



Two spheres are said to cut orthogonally if the tangent planes at a point of intersection are at right angles.

The radii of such spheres through their point of intersection P, being ⊥ to the tangent planes at P are also at right angles. Thus two spheres cut orthogonally, if the square of the distance between their centre

= sum of the squares of their radii.

(Art) Show that the condition for spheres

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

and $x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$

to cut orthogonally is $2uu'+2vv'+2ww'=d+d'$

Proof: The centres of the spheres are

$$C(-u,-v,-w),\ C'(-u',-v',-w')$$

and their radii are

$$r=\sqrt{u^2+v^2+w^2-d}$$

$$r'=\sqrt{u'^2+v'^2+w'^2-d'}$$

Now these spheres will cut orthogonally, if

$$(CC')^2=r^2+r'^2$$

or $$(u-u')^2+(v-v')^2+(w-w'^2)$$

$$=u^2+v^2+w^2-d+u'^2+v'^2+w'^2-d$$

or $2uu'+2vv'+2ww'=d+d$ is the required condition.

— x —

(Art) Find the condition that the plane $lx+my+nz=p$ is a tangent to the sphere $x^2+y^2+z^2=a^2$.

Proof: Eqn of the sphere is $x^2+y^2+z^2=a^2$ — ①

Let eqn of the plane be $lx+my+nz=p$ — ② touch the sphere ① at the point (x_1,y_1,z_1). Then the eqn of the tangent plane to ① at (x_1,y_1,z_1) is

$$xx_1+yy_1+zz_1=a^2 \quad \text{— ③}$$

Comparing ② & ③ $\dfrac{l}{x_1}=\dfrac{m}{y_1}=\dfrac{n}{z_1}=p/a^2$

$\Rightarrow x_1=\dfrac{a^2l}{p},\ y_1=\dfrac{a^2m}{p},\ z_1=\dfrac{a^2n}{p}$

Since (x_1,y_1,z_1) is on ① so

$$x_1^2+y_1^2+z_1^2=a^2$$

$\therefore \dfrac{a^4l^2}{p^2}+\dfrac{a^4m^2}{p^2}+\dfrac{a^4n^2}{p^2}=a^2$

$\Rightarrow a^2(a^2l^2+a^2m^2+a^2n^2)=p^2$

$\therefore a^2(l^2+m^2+n^2)=p^2$ is the required condition