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Degree:- 1 (H+S)

**Chapter:-
Sphere**

**Topic:- Solid
geometry (3D)**

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Problems of Sphere

(7)

Ex¹ Find the equation of the sphere through the points $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ and $(1,2,3)$. Locate its centre and find the radius.

Solⁿ: General equation of the sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ — (1)

(1) passes through $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ and $(1,2,3)$.
So $d = 0$ — (2)

$$0 + 1 + 1 + 2v - 2w + d = 0 \Rightarrow v - w + 1 = 0 \text{ — (3)}$$

$$1 + 4 + 2u + 4v + d = 0 \Rightarrow -2u + 4v + 5 = 0 \text{ — (4)}$$

$$1 + 4 + 9 + 2u + 4v + 6w + d = 0 \Rightarrow u + 2v + 3w + 7 = 0 \text{ — (5)}$$

Solving (3), (4) and (5) we get the value of u, v, w

$$(2) + (4) \Rightarrow -2u + 5v - w + 6 = 0$$

$$2 \times (5) \Rightarrow 2u + 4v + 6w + 14 = 0$$

$$\hline 9v + 5w + 20 = 0$$

$$9 \times (3) \quad 9v - 9w + 9 = 0$$

$$\hline 14w = -11 \Rightarrow w = -\frac{11}{14} \checkmark$$

$$\text{From (3)} \quad v = w - 1 = -\frac{11}{14} - 1 = -\frac{25}{14} \checkmark$$

$$\text{From (4)} \quad -2u + \cancel{2} \left(-\frac{25}{14} \right) + 5 = 0$$

$$\text{or } -2u = \frac{50}{7} - 5 = \frac{50 - 35}{7} = \frac{15}{7} \Rightarrow u = -\frac{15}{14} \checkmark$$

Putting the value of u, v, w, d in (1)

$$x^2 + y^2 + z^2 + \cancel{2} \left(-\frac{15}{14} \right) x + \cancel{2} \left(-\frac{25}{14} \right) y + \cancel{2} \left(-\frac{11}{14} \right) z + 0 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0 \text{ — (6)}$$

which is the required eqⁿ of the sphere.

Its centre is $\left(\frac{15}{14}, \frac{25}{14}, -\frac{11}{14} \right)$ or $\boxed{(-u, -v, -w)}$

and the radius = $\left(-\frac{15}{14} \right)^2 + \left(-\frac{25}{14} \right)^2 + \left(-\frac{11}{14} \right)^2 = \sqrt{\frac{971}{14}}$ or

② Deduce the eqn of the sphere described on the line joining the points $(2, -1, 4)$ and $(-2, 2, -2)$ as diameter. Find the area of the circle in which the sphere is intersected by the plane $2x + y - z = 3$.

Soln:- Equation of the sphere is
 $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$
 Here, $x_1 = 2, y_1 = -1, z_1 = 4$

$\Rightarrow (x-2)(x+2) + (y+1)(y-2) + (z-4)(z+2) = 0$ $x_2 = -2, y_2 = 2, z_2 = -2$

$\Rightarrow x^2 + y^2 + z^2 - y - 2z - 14 = 0$ — (1)

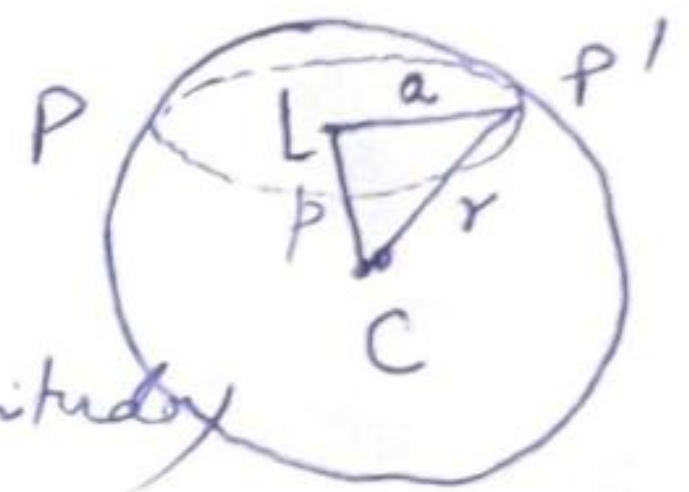
Its centre is $C(0, -1/2, 1)$

radius $r = \sqrt{0 + \frac{1}{4} + 1 + 14} = \sqrt{6\frac{1}{4}}$

Let the given plane is $2x + y - z - 3 = 0$ — (2)
 cut the sphere (1) in the circle PP' having centre L .

$p =$ perpendicular CL from C on the plane (2) $= \frac{\frac{1}{2} - 1 - 3}{\sqrt{4+1+1}} = \frac{-\frac{5}{2}}{2\sqrt{6}}$

(in magnitude)



If a be the radius of the circle PP' , then

$a^2 = r^2 - p^2 = \frac{6\frac{1}{4}}{4} - \frac{49}{24} = \frac{312}{24}$

Hence the area of circle $PP' = \pi a^2 = \frac{312}{24} \pi$

③ A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C . Show that the locus of the centre of the sphere $OABC$ is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.

Soln:- Let the centre of the sphere $OABC$ be $P(f, g, h)$ so that its radius $OP = \sqrt{f^2 + g^2 + h^2}$

The eqn of the sphere is $(x-f)^2 + (y-g)^2 + (z-h)^2 = f^2 + g^2 + h^2$

$\Rightarrow x^2 + y^2 + z^2 - 2fx - 2gy - 2hz = 0$ — (1)

To find OA , put $y=0, z=0$ in (1) $\Rightarrow x^2 - 2fx = 0$ i.e. $OA = x = 2f$.
 Similarly, $OB = 2g, OC = 2h$.

Thus the eqn of the plane ABC is $\frac{x}{2f} + \frac{y}{2g} + \frac{z}{2h} = 1$

Since the plane passes through $(a, b, c) \Rightarrow \frac{a}{2f} + \frac{b}{2g} + \frac{c}{2h} = 1$

Hence the locus of the centre (f, g, h) of the sphere is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

- ④ Find the equation of the sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$, $x + y + z = 3$ as a great circle. ⑨

Soln: The eqn of any sphere through the given circle is

$$x^2 + y^2 + z^2 + 10y - 4z - 8 + k(x + y + z - 3) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + kx + (10+k)y + (-4+k)z + (-8-3k) = 0 \quad \text{--- ①}$$

In order that ① may have the given circle as its great circle, its centre $\left[-\frac{k}{2}, -\frac{(10+k)}{2}, \frac{(4-k)}{2}\right]$ must

lie on the plane $x + y + z = 3$

$$-\frac{k}{2} - \frac{10+k}{2} + \frac{4-k}{2} = 3, \Rightarrow k = -4$$

whence ① becomes, $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

which is the required equation.

- ⑤ Find the equation of the smallest sphere which contains the circle $x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 = 0$ and $2x + 2y + z + 1 = 0$.

Soln: Equation of any sphere containing the given circle is

$$x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 + k(2x + 2y + z + 1) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + (2+2k)x + (6+2k)y + (4+k)z + (-11+k) = 0 \quad \text{--- ①}$$

$$\text{Its radius, } r^2 = (1+k)^2 + (3+k)^2 + \left(2 + \frac{1}{2}k\right)^2 - (k-11)$$

$$= \frac{9}{4} \left[(k+2)^2 + \frac{64}{9} \right]$$

Now r^2 has the least value when $k = -2$

Putting the value of k in ①

$$x^2 + y^2 + z^2 - 2x + 2y + 2z - 13 = 0$$

which is the required smallest sphere.

⑥ Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$ and $x + 2y - 7z = 0$ lie on the same sphere and find its equation. (10)

Sol: Eqn of any sphere containing the first circle is

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + k(5y + 6z + 1) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + (-2x) + (3+5k)y + (4+6k)z + (k-5) = 0 \quad \text{--- (1)}$$

similarly eqn of any sphere containing the second given circle is

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + k'(x + 2y - 7z) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + (-3+k')x + (-4+2k')y + (5-7k')z - 6 = 0 \quad \text{--- (2)}$$

① and ② will represent the same sphere when

$$-2 = -3 + k' \Rightarrow k' = 1 \quad \text{--- (3)}$$

$$3 + 5k = -4 + 2k' \quad \text{--- (4)}$$

$$4 + 6k = 5 - 7k' \quad \text{--- (5)}$$

$$k - 5 = -6 \quad \text{--- (6)} \Rightarrow k = -1 \quad \text{--- (6)}$$

Clearly $k = -1$ and $k' = 1$ also satisfy (4) and (5).

This shows that the given circles lie on the same sphere.

Putting $k = -1$ in ① or $k' = 1$ in ②, we get

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$$

which is the desired sphere.

⑦ Find the eqns of the sphere passing through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0$, $y = 0$ and touching the plane $3y + 4z + 5 = 0$.

Sol: Eqn of any sphere through the given circle is

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + ky = 0 \Rightarrow x^2 + y^2 + z^2 - 6x + ky - 2z + 5 = 0$$

$$\therefore \text{Its Centre is } \left(3, -\frac{k}{2}, 1 \right) \text{ and radius} = \sqrt{9 + \frac{k^2}{4} + 1 - 5} = \sqrt{5 + \frac{k^2}{4}}$$

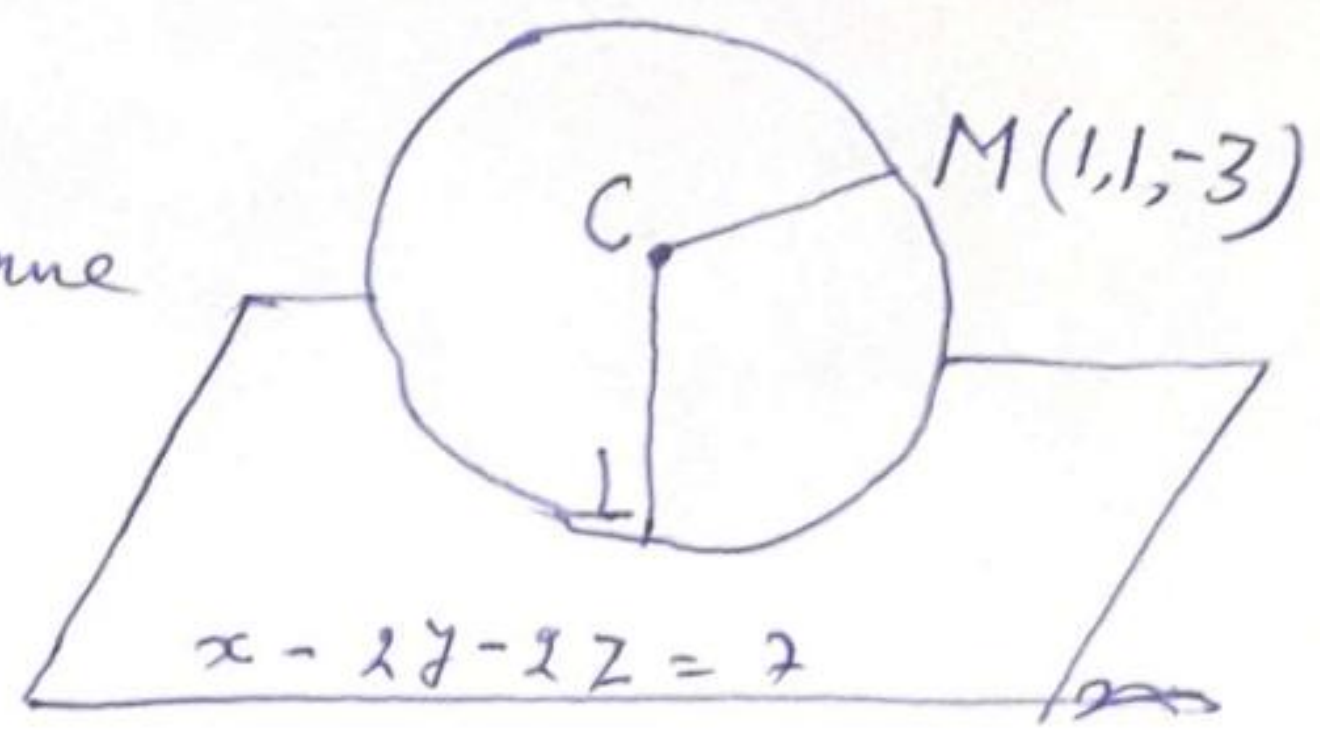
The sphere ① will touch the plane $3y + 4z + 5 = 0$ if distance of the centre $\left(3, -\frac{k}{2}, 1 \right)$ from the plane = radius.

$$\therefore \frac{3\left(-\frac{k}{2}\right) + 4 + 5}{\sqrt{9 + 16}} = \sqrt{5 + \frac{k^2}{4}} \Rightarrow 4k^2 + 27k + 44 = 0 \Rightarrow k = -\frac{11}{2} \text{ or } -4.$$

Putting the value of k in ①, $x^2 + y^2 + z^2 - 6x - \frac{11}{4}y + 2z + 5 = 0$ and $x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0$ as the two required spheres.

⑧ Find the equation of the sphere which touches the plane $x - 2y - 2z = 7$ at the point $L(3, -1, -1)$ and passes through the point $M(1, 1, -3)$.

Soln: If C is the centre of the sphere, then CL is perpendicular to the given plane $x - 2y - 2z = 7$.



∴ The dir's of CL being $\{1, -2, -2\}$, the eqn of CL is

$$\frac{x-3}{1} = \frac{y+1}{-2} = \frac{z+1}{-2} = k \text{ (say)}$$

Any point on CL is $(k+3, -2k-1, -2k-1)$ which will represent C for some value of k .

Since M lies on the sphere, therefore its radius $CL = CM$

$$or (CL)^2 = (CM)^2$$

$$(k+3-3)^2 + (-2k-1+1)^2 + (-2k-1+1)^2 = (k+3-1)^2 + (-2k-1-1)^2 + (-2k-1+3)^2$$

$$or 4k = -12 \Rightarrow k = -3$$

∴ The centre C is $(0, 5, 5)$ and radius $CL = \sqrt{9+36+36} = 9$

Hence the required equation of the sphere is

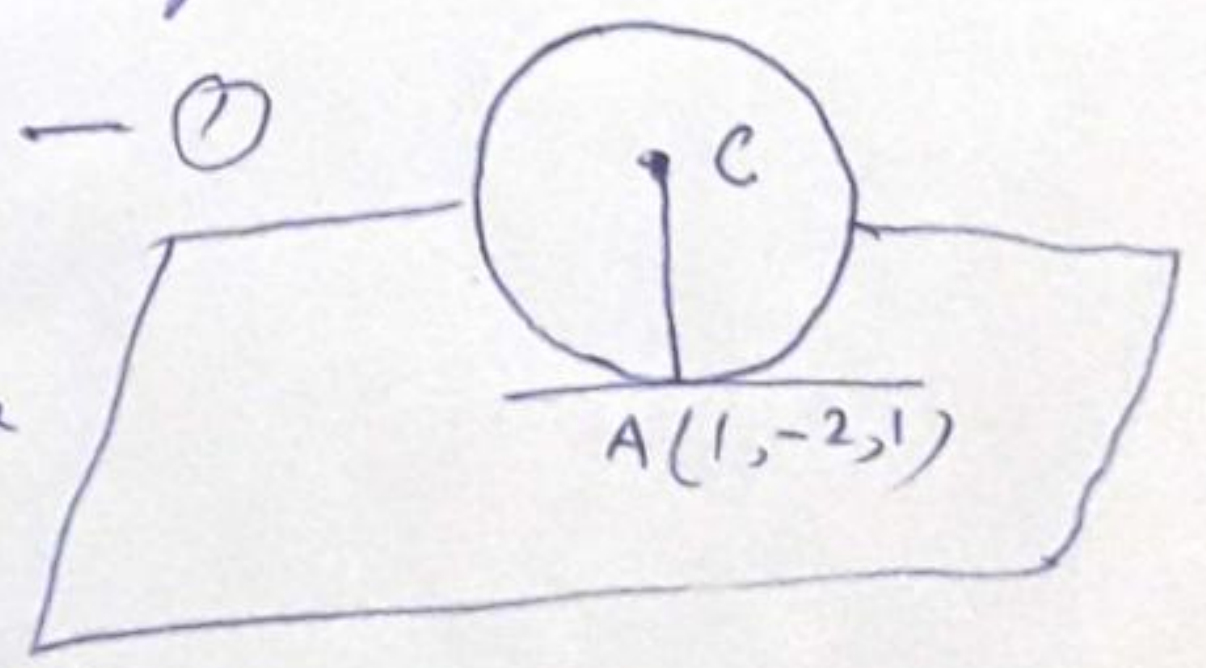
$$(x-0)^2 + (y-5)^2 + (z-5)^2 = (9)^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$$

⑨ Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts the sphere $R^2 - 2(2I - 3J) \cdot R + 4 = 0$ orthogonally.

Soln: Given plane $3x + 2y - z + 2 = 0$ — (i)

will touch the required sphere at $A(1, -2, 1)$ if its centre lies on the normal to (i) at A .



The equations of the normal to (i) at A are

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1}$$

Any point on this line is $C(3r+1, 2r-2, -r+1)$

Also radius AC of the required sphere.

$$= \sqrt{(3r)^2 + (2r)^2 + (-r)^2} = r\sqrt{14}$$

Since the required sphere cuts the given sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0 \text{ orthogonally,}$$

$$\begin{cases} \text{Centre} = (2, -3, 0) \\ \text{radius} = 3 \end{cases}$$

therefore (distance between their centres)²

$$= \sum \text{of squares of their radii}$$

$$\text{i.e. } (3r+1-2)^2 + (2r-2+3)^2 + (-r+1)^2 = 14r^2 + 9$$

$$\text{or } r = -3/2.$$

Thus centre C is $(-\frac{7}{2}, 5, \frac{5}{2})$ and radius = $\frac{3\sqrt{14}}{2}$

Hence the required sphere is

$$\left(x + \frac{7}{2}\right)^2 + (y+5)^2 + \left(z - \frac{5}{2}\right)^2 = \left(\frac{3\sqrt{14}}{2}\right)^2$$

$$\Rightarrow x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0 \quad \checkmark$$