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Chapter:-
Sphere

Topic:- Solid geometry (3D)

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Problems of sphere

Ex1 Find the equation of the sphere through the points $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ and $(1,2,3)$. Locate its centre and find the radius.

Soln: General equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

(1) passes through $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ and $(1,2,3)$.

so $d=0$ --- (2)

$0+1+1+2v-2w+d=0 \Rightarrow v-w+1=0$ --- (3)

$1+4-2u+4v+d=0 \Rightarrow -2u+4v+5=0$ --- (4)

$1+4+9+2u+4v+6w+d=0 \Rightarrow u+2v+3w+7=0$ --- (5)

solving (3), (4) and (5) we get the value of u, v, w

(3) + (4) $\Rightarrow$ $-2u+5v-w+6=0$

$2\times$(5) $\Rightarrow$ $2u+4v+6w+14=0$

$9v+5w+20=0$

$9\times$(3) $\quad 9v-9w+9=0$

$\quad -\quad +\quad -$

$14w=-11 \Rightarrow w=-\frac{11}{14}$ ✓

From (3) $v=w-1=-\frac{11}{14}-1=\frac{-25}{14}$ ✓

From (4) $-2u+\cancel{4}^{2}\left(-\frac{25}{\cancel{14}_{7}}\right)+5=0$

or $-2u=\frac{50}{7}-5=\frac{50-35}{7}=\frac{15}{7} \Rightarrow u=-\frac{15}{14}$ ✓

Putting the value of u, v, w, d in (1)

$$x^2+y^2+z^2+\cancel{2}\left(-\frac{15}{\cancel{14}_7}\right)x+\cancel{2}\left(-\frac{25}{\cancel{14}_7}\right)y+\cancel{2}\left(-\frac{11}{\cancel{14}_7}\right)z+0=0$$

$$\Rightarrow x^2+y^2+z^2-\frac{15}{7}x-\frac{25}{7}y-\frac{11}{7}z=0 \quad \text{--- (6) Ans}$$

which is the required eqn of the sphere.

Its centre is $\left(\frac{15}{14}, \frac{25}{14}, -\frac{11}{14}\right)$ Ans $\boxed{(-u,-v,-w)}$

and the radius $= \left(-\frac{15}{14}\right)^2+\left(-\frac{25}{14}\right)^2+\left(\frac{-11}{14}\right)^2=\sqrt{\frac{971}{14}}$ Ans

(2) Deduce the eqn of the sphere described on the line joining the points $(2,-1,4)$ and $(-2,2,-2)$ as diameter. Find the area of the circle in which the sphere is intersected by the plane $2x+y-z=3$.

Soln:- Equation of the sphere is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

Here, $x_1=2,\ y_1=-1,\ z_1=4$

$x_2=-2,\ y_2=2,\ z_2=-2$

$$\Rightarrow (x-2)(x+2)+(y+1)(y-2)+(z-4)(z+2)=0$$

$$\Rightarrow x^2+y^2+z^2-y-2z-14=0 \quad \text{---} \ (1)$$

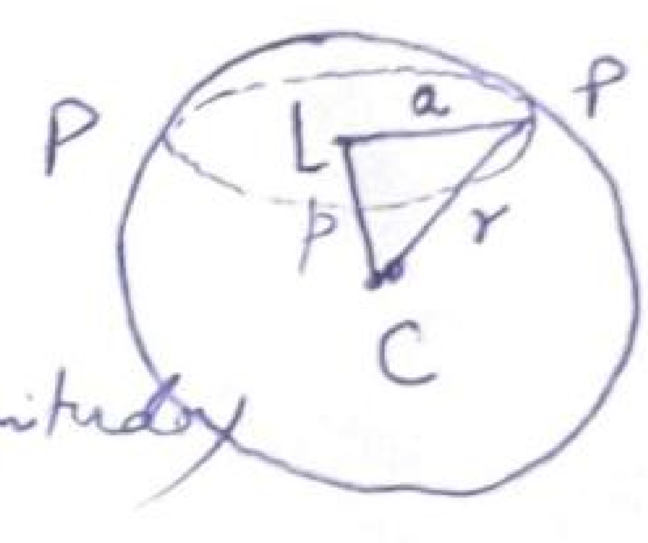
Its centre is $C\left(0,-\tfrac{1}{2},1\right)$

radius $r=\sqrt{0+\frac{1}{4}+1+14}=\sqrt{61/4}$

Let the given plane is $2x+y-z-3=0$ — (2)

cut the sphere (1) in the circle PP' having centre L.

$p=$ perpendicular CL from C on the plane (2) $=\dfrac{\frac{1}{2}-1-3}{\sqrt{4+1+1}}=\dfrac{7}{2\sqrt{6}}$ (in magnitude)



If a be the radius of the circle PP', then

$$a^2=r^2-p^2=\frac{61}{4}-\frac{49}{24}=\frac{317}{24}$$

Hence the area of circle $PP'=\pi a^2=\dfrac{317}{24}\pi$.

(3) A plane passes through a fixed point (a,b,c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is $a/x+b/y+c/z=2$.

Soln: Let the centre of the sphere $OABC$ be $P(f,g,h)$ so that its radius $OP=\sqrt{f^2+g^2+h^2}$

The eqn of the sphere is $(x-f)^2+(y-g)^2+(z-h)^2=f^2+g^2+h^2$

$$\Rightarrow x^2+y^2+z^2-2fx-2gy-2hz=0 \quad \text{---} \ (1)$$

To find OA, put $y=0, z=0$ in (1) $\Rightarrow x^2-2fx=0$ i.e. $OA=x=2f$.

Similarly, $OB=2g,\ OC=2h$.

Thus the eqn of the plane ABC is $\dfrac{x}{2f}+\dfrac{y}{2g}+\dfrac{z}{2h}=1$

Since the plane passes through (a,b,c) $\Rightarrow \dfrac{a}{2f}+\dfrac{b}{2g}+\dfrac{c}{2h}=1$

Hence the locus of the centre (f,g,h) of the sphere is

$a/x+b/y+c/z=2$

(4) Find the equation of the sphere having the circle
$$x^2+y^2+z^2+10y-4z-8=0,\ x+y+z=3$$
as a great circle.

Sol^n: The eq^n of any sphere through the given circle is
$$x^2+y^2+z^2+10y-4z-8+k(x+y+z-3)=0$$
$$\Rightarrow x^2+y^2+z^2+kx+(10+k)y+(-4+k)z+(-8-3k)=0 \quad \text{—(1)}$$

In order that (1) may have the given circle as its great circle, its centre $\left[-\frac{k}{2},\ -\frac{(10+k)}{2},\ \frac{(4-k)}{2}\right]$ must lie on the plane $x+y+z=3$
$$-\frac{k}{2}-\frac{10+k}{2}+\frac{4-k}{2}=3,\ \Rightarrow k=-4$$
whence (1) becomes, $x^2+y^2+z^2-4x+6y-8z+4=0$
which is the required equation.

(5) Find the equation of the smallest sphere which contains the circle $x^2+y^2+z^2+2x+6y+4z-11=0$ and $2x+2y+z+1=0$.

Sol^n: Equation of any sphere containing the given circle is
$$x^2+y^2+z^2+2x+6y+4z-11+k(2x+2y+z+1)=0$$
$$\Rightarrow x^2+y^2+z^2+(2+2k)x+(6+2k)y+(4+k)z+(-11+k)=0 \quad \text{—(1)}$$

Its radius, $r^2=(1+k)^2+(3+k)^2+(2+\frac{1}{2}k)^2-(k-11)$
$$=\frac{9}{4}\left[(k+2)^2+\frac{64}{9}\right]$$

Now r^2 has the least value when $k=-2$

Putting the value of k in (1)
$$x^2+y^2+z^2-2x+2y+2z-13=0$$
which is the required smallest sphere.

(6) Prove that the circles $x^2+y^2+z^2-2x+3y+4z-5=0$, $5y+6z+1=0$ and $x^2+y^2+z^2-3x-4y+5z-6=0$, $x+2y-7z=0$ lie on the same sphere and find its equation.

Soln: Eqn of any sphere containing the first circle is

$$x^2+y^2+z^2-2x+3y+4z-5+K(5y+6z+1)=0$$

$$\Rightarrow x^2+y^2+z^2+(-2x)+(3+5K)y+(4+6K)z+(K-5)=0 \quad \text{--- (1)}$$

similarly eqn of any sphere containing the second given circle is

$$x^2+y^2+z^2-3x-4y+5z-6+K'(x+2y-7z)=0$$

$$\Rightarrow x^2+y^2+z^2+(-3+K')x+(-4+2K')y+(5-7K')z-6=0 \quad \text{--- (2)}$$

(1) and (2) will represent the same sphere when

$$-2=-3+K' \Rightarrow K'=1 \quad \text{--- (3)}$$

$$3+5K=-4+2K' \quad \text{--- (4)}$$

$$4+6K=5-7K' \quad \text{--- (5)}$$

$$K-5=-6 \quad \text{---} \Rightarrow K=-1 \quad \text{--- (6)}$$

Clearly $K=-1$ and $K'=1$ also satisfy (4) and (5).

This shows that the given circles lie on the same sphere.

Putting $K=-1$ in (1) or $K'=1$ in (2), we get

$$x^2+y^2+z^2-2x-2y-2z-6=0$$

which is the desired sphere.

(7) Find the eqns of the sphere passing through the circle $x^2+y^2+z^2-6x-2z+5=0$, $y=0$ and touching the plane $3y+4z+5=0$.

Soln: Eqn of any sphere through the given circle is

$$x^2+y^2+z^2-6x-2z+5+Ky=0 \Rightarrow x^2+y^2+z^2-6x+Ky-2z+5=0$$

$\therefore$ Its centre is $\left(3,-\frac{K}{2},1\right)$ and radius $=\sqrt{9+\frac{K^2}{4}+1-5}=\sqrt{5+\frac{K^2}{4}}$

The sphere (1) will touch the plane $3y+4z+5=0$ if

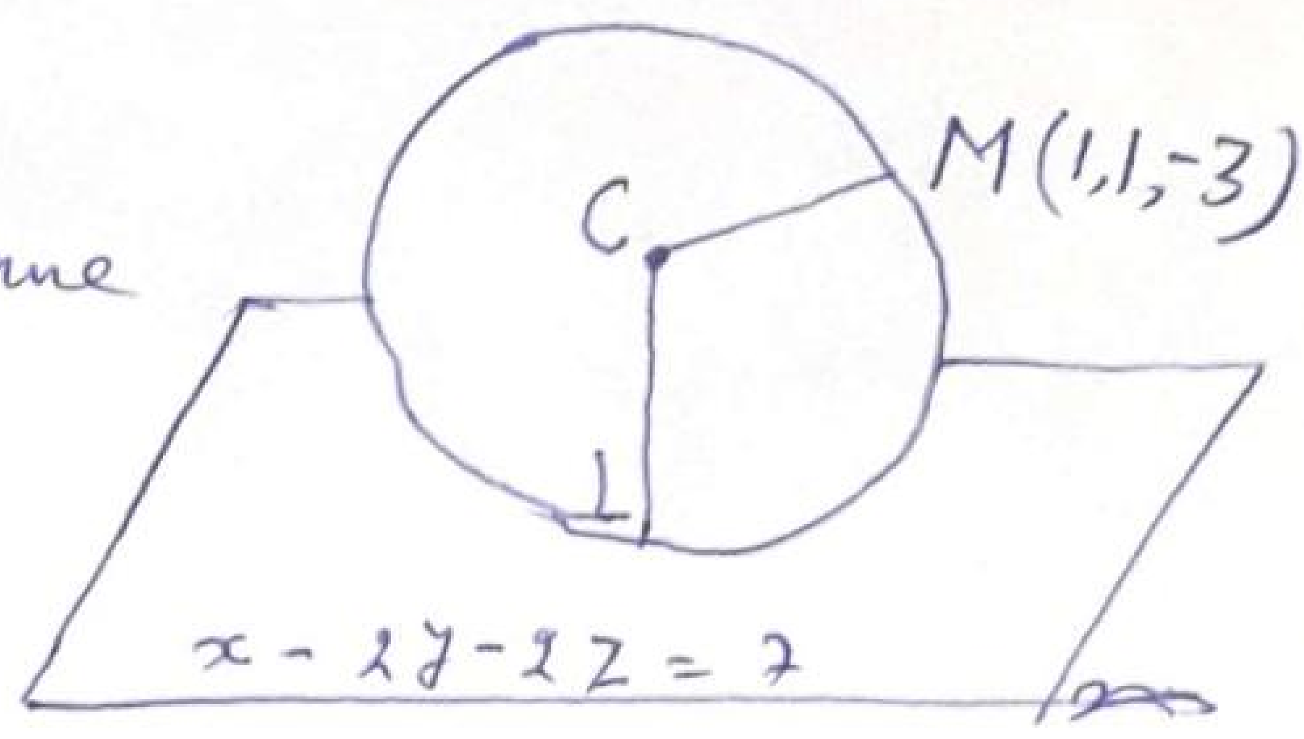
+ distance of the centre $\left(3,-\frac{K}{2},1\right)$ from the plane = radius.

i.e. $$\frac{3\left(-\frac{K}{2}\right)+4+5}{\sqrt{9+16}}=\sqrt{5+\frac{K^2}{4}} \Rightarrow 4K^2+27K+44=0 \Rightarrow K=\frac{-11}{2} \text{ or } -4.$$

Putting the value of K in (1), $x^2+y^2+z^2-6x-\frac{11}{4}y+2z+5=0$ and

$x^2+y^2+z^2-6x-4y-2z+5=0$ as the two required spheres.

(8) Find the equation of the sphere which touches the plane $x-2y-2z=7$ at the point $L(3,-1,-1)$ and passes through the point $M(1,1,-3)$.



Soln: If C is the centre of the sphere, then CL is perpendicular to the given plane $x-2y-2z=7$.

$\therefore$ The d.r.s of CL being $1,-2,-2$, the eqn of CL is

$$\frac{x-3}{1}=\frac{y+1}{-2}=\frac{z+1}{-2}=k \text{ (say)}$$

Any point on CL is $(k+3,-2k-1,-2k-1)$ which will represent C for some value of k.

Since M lies on the sphere, therefore its radius $CL=CM$

or $(CL)^2=(CM)^2$

$$(k+3-3)^2+(-2k-1+1)^2+(-2k-1+1)^2$$
$$=(k+3-1)^2+(-2k-1-1)^2+(-2k-1+3)^2$$

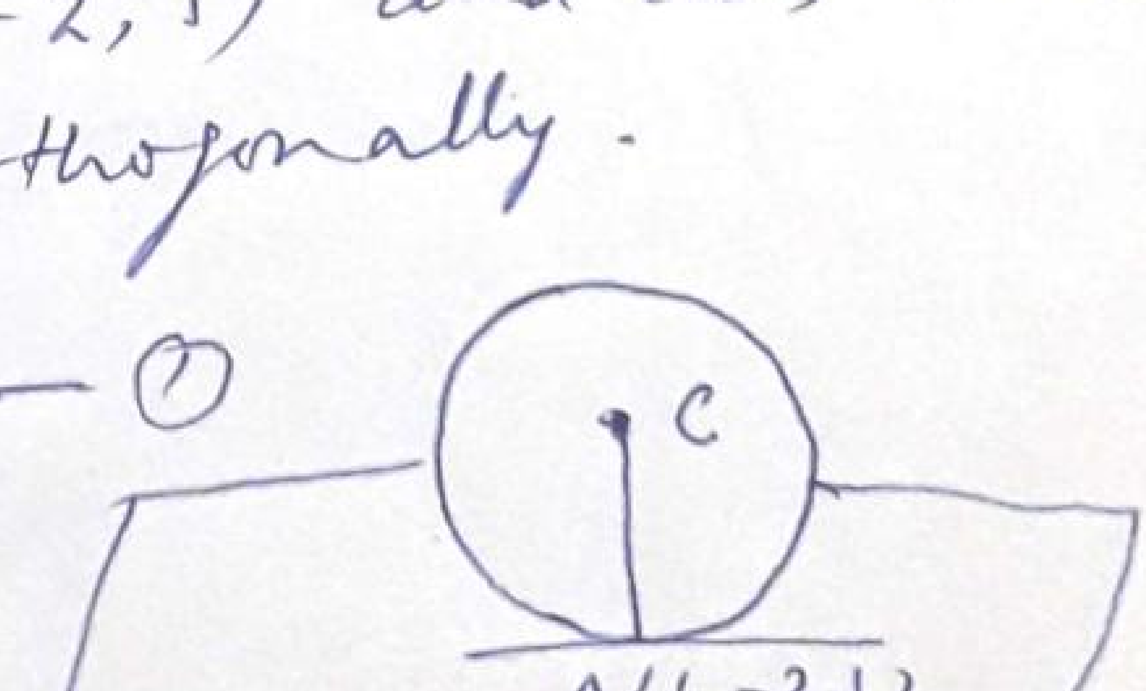
or $4k=-12 \Rightarrow k=-3$

$\therefore$ The Centre C is $(0,5,5)$ and radius $CL=\sqrt{9+36+36}=9$

Hence the required equation of the sphere is

$$(x-0)^2+(y-5)^2+(z-5)^2=(9)^2$$
$$\Rightarrow x^2+y^2+z^2-10y-10z-31=0$$

(9) Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts the sphere $R^2-2(2I-3J)\cdot R+4=0$ orthogonally.



Soln: Given plane $3x+2y-z+2=0$ — (1) will touch the required sphere at $A(1,-2,1)$ if its Centre lies on the normal to (1) at A.

The equations of the normal to (i) at A are

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z-1}{-1}.$$

Any point on this line is $C\ (3r+1, 2r-2, -r+1)$

Also radius AC of the required sphere.

$$= \sqrt{(3r)^2+(2r)^2+(-r)^2} = r\sqrt{14}$$

Since the required sphere cuts the given sphere

$x^2+y^2+z^2-4x+6y+4=0$ orthogonally.

$$\begin{cases} \text{centre} = (2,-3,0) \\ \text{radius} = 3 \end{cases}$$

therefore (distance between their centres)^2

$= \sum$ of squares of their radii

i.e. $(3r+1-2)^2+(2r-2+3)^2+(-r+1)^2 = 14r^2+9$

or $r = -3/2$.

Thus centre C is $\left(-\frac{7}{2}, 5, \frac{5}{2}\right)$ and radius $= \frac{3\sqrt{14}}{2}$

Hence the required sphere is

$$\left(x+\frac{7}{2}\right)^2+(y+5)^2+\left(z-\frac{5}{2}\right)^2 = \left(3\sqrt{14}/2\right)^2$$

$$\Rightarrow x^2+y^2+z^2+7x+10y-5z+12=0$$

If the tangent plane to the sphere $x^2+y^2+z^2=r^2$ makes intercepts a, b, c on the co-ordinate axes, prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$

Soln:- The eqn of the sphere is $x^2+y^2+z^2=r^2$ —— (1)

The eqn of the tangent plane to the sphere at (x_1,y_1,z_1) is

$$xx_1+yy_1+zz_1=r^2 \text{ —— (2)}$$

This tangent plane makes intercepts a, b, c on the co-ordinate axes. Therefore $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ must satisfy (2).

$\therefore ax_1=r^2 \Rightarrow x_1=r^2/a$ similarly $y_1=r^2/b$ and $z_1=r^2/c$.

The point (x_1,y_1,z_1) also lies on (1).

$$\therefore x_1^2+y_1^2+z_1^2=r^2 \Rightarrow \frac{r^4}{a^2}+\frac{r^4}{b^2}+\frac{r^4}{c^2}=r^2 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$$

(11) Find the equations of spheres which pass through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$ and touch the plane $4x+3y=15$.

Soln:- Let the eqn of the sphere through the given circle $x^2+y^2+z^2=5$, $x+2y+3z=3$ be

$$x^2+y^2+z^2-5+k(x+2y+3z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+kx+2ky+3kz+(-3k-5)=0 \text{ —— (1)}$$

comparing it

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

Here $2u=k \Rightarrow u=\frac{k}{2}$

$2v=2k \Rightarrow v=k$

$2w=3k \Rightarrow w=\frac{3k}{2}$

$d=\text{const}=-3k-5$

$\therefore$ Centre $=(-u,-v,-w)$

$=\left(-\frac{k}{2},-k,-\frac{3k}{2}\right)$

radius $=\sqrt{u^2+v^2+w^2-d}$

$=\sqrt{\frac{k^2}{4}+\frac{k^2}{1}+\frac{9k^2}{4}+3k+5}$

Since the sphere (1) touches the plane (2) $4x+3y=15$, therefore the length of the perpendicular drawn from the centre of the sphere to this plane must be equal to the radius of the sphere.

$$\therefore \frac{4\left(-\frac{k}{2}\right)+3(-k)-15}{\sqrt{16+9}}=\sqrt{\frac{14k^2+12k+20}{4}}$$

$\Rightarrow 4(k^2+6k+9)=14k^2+12k+20$

$\Rightarrow k=2$ or $-4/5$

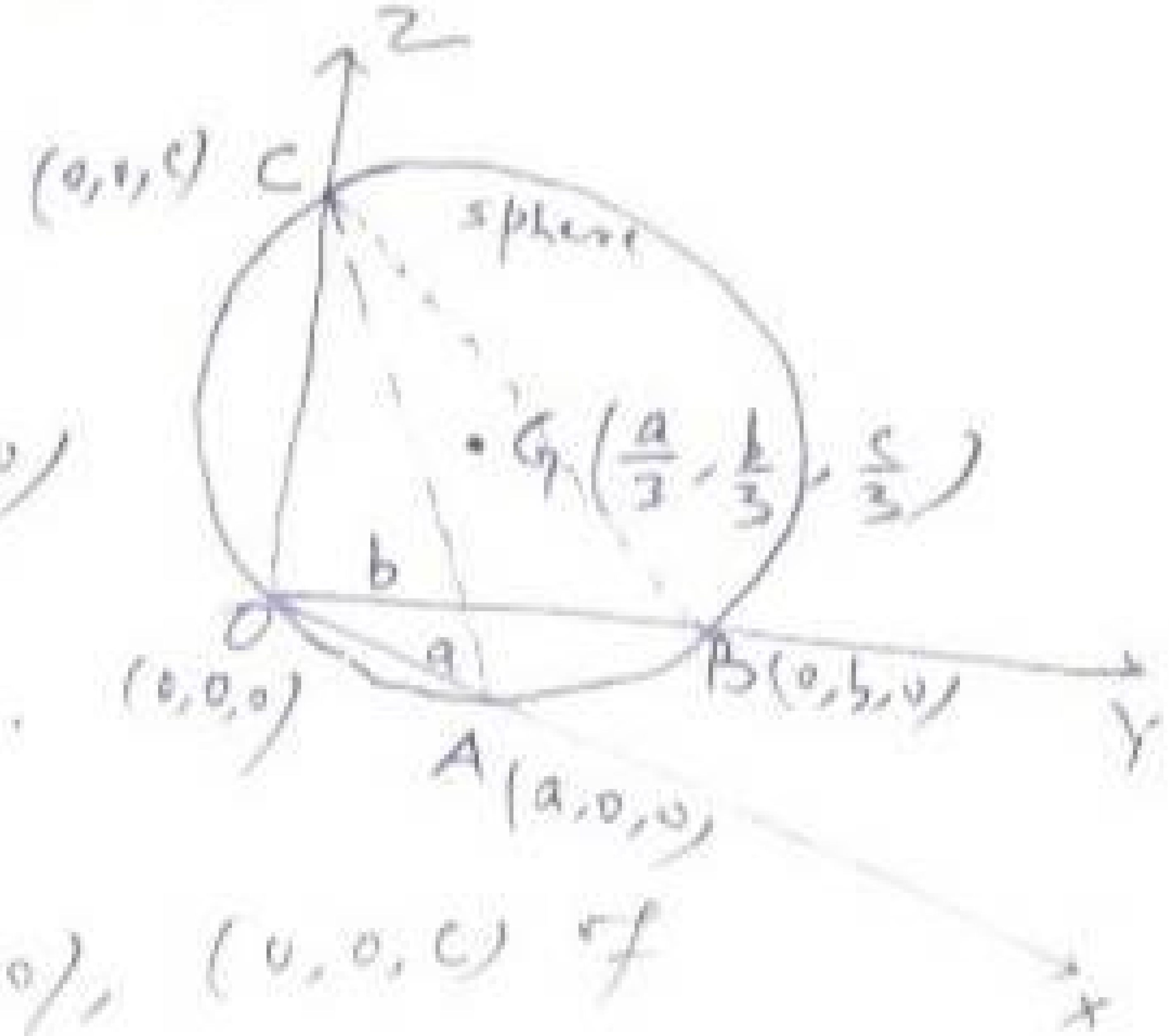
Putting the value of k in (1)

$x^2+y^2+z^2+2x+4y+6z-11=0$

and $5(x^2+y^2+z^2)-4x-8y-12z-13=0$

are required eqns of the sphere

(12) A sphere of constant radius K passes through the origin and meets the axes in A, B, C. Prove that the centroid of the ΔABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.



Sol^n:- Let the sphere of constant radius K be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (1)

Since it passes through the origin (0,0,0)

$\therefore d=0$

This sphere cuts the axes at A, B, C.

Let $OA=a$, $OB=b$, $OC=c$.

Then the co-ordinates $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ of A, B, C must satisfy (1)

$\therefore a^2+2ua=0 \Rightarrow 2u=-a$, $2v=-b$, $2w=-c$

The radius of the sphere (1) $= \sqrt{u^2+v^2+w^2-d} = \sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}} = k$

$\Rightarrow a^2+b^2+c^2 = 4k^2$ —— (2)

The centroid of the ΔABC is $\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$

$= \left(a/3, b/3, c/3\right)$.

Let the centroid of the ΔABC be (x_1, y_1, z_1)

Then $x_1=\frac{a}{3}$, $y_1=b/3$, $z_1=c/3 \Rightarrow \left.\begin{array}{l} a=3x_1 \\ b=3y_1 \\ c=3z_1 \end{array}\right\}$

Putting the value of a, b, c in (2)

$9(x_1^2+y_1^2+z_1^2)=4k^2$

Hence the locus of (x_1, y_1, z_1) is $9(x^2+y^2+z^2)=4k^2$.

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