Degree:–1(H+S)

Chapter:– Area of curves length determined from Polar Equation

Topic:– Integral Calculus

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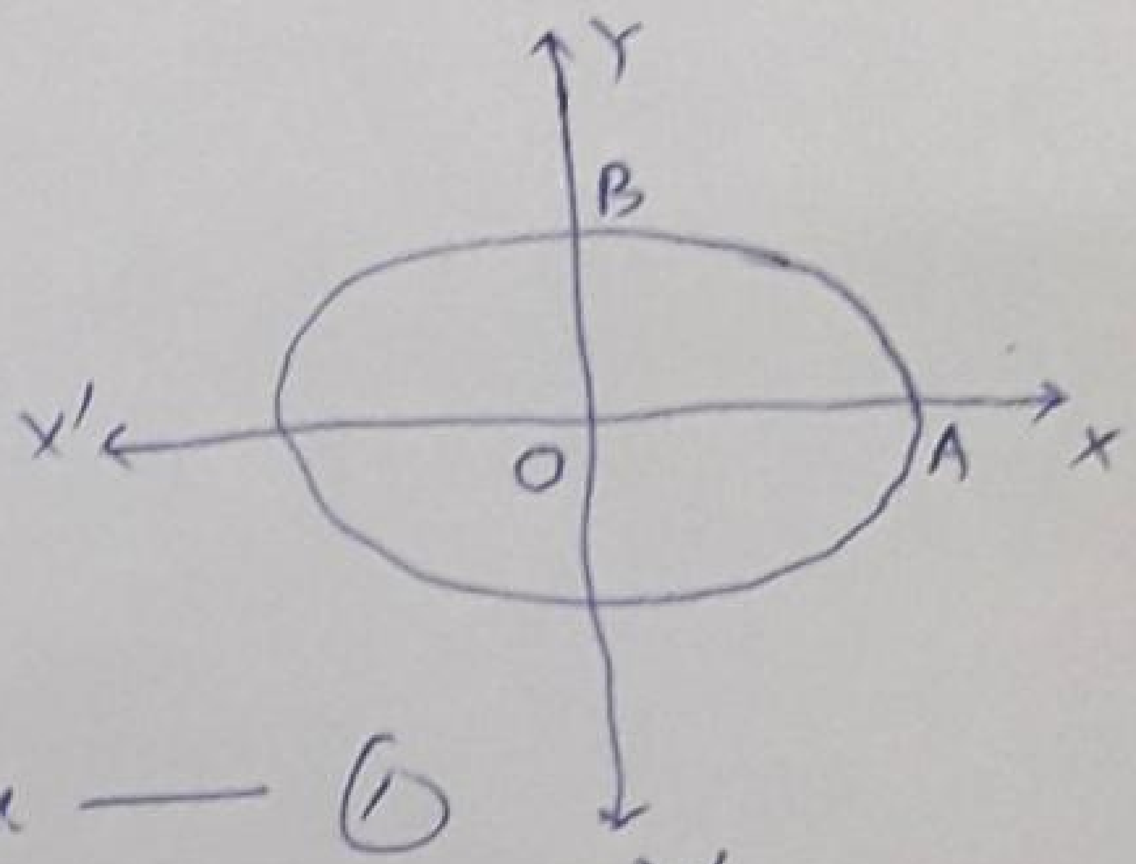
Area of curves

(1) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Soln:- An ellipse is symmetrical about both the axes.

$\therefore$ Area of the ellipse = 4 × area of OAB

extent of OAB along x-axis is $O(x=0)$ to $A(x=a)$

$\therefore$ Required area of the ellipse $= 4\int_0^a y\,dx$ — (1)

Put, $x = a\cos\theta \Rightarrow dx = -a\sin\theta\, d\theta$

$\left.\begin{array}{l} x=0 \Rightarrow \cos\theta = 0 = \cos\frac{\pi}{2} \Rightarrow \theta = \pi/2 \\ x = a \Rightarrow \cos\theta = 1 = \cos 0^\circ \Rightarrow \theta = 0^\circ \end{array}\right\}$

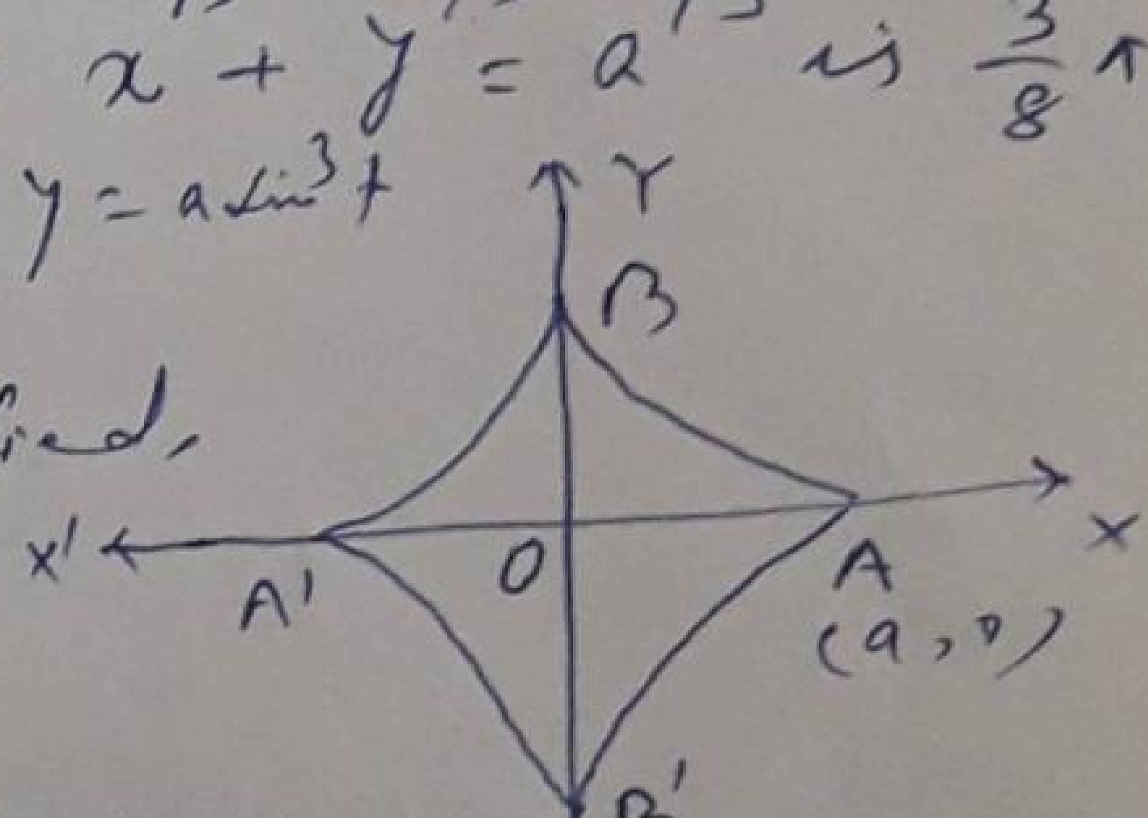
and $y = b\sin\theta$

$\therefore\; A = 4\int_0^a b\sin\theta\,(-a\sin\theta\, d\theta) = +4ab\int_0^{\pi/2}\sin^2\theta\, d\theta$

$= 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta = \cancel{4}ab\times\frac{1}{\cancel{2}}\times\frac{\pi}{\cancel{2}} = \pi ab.$

(2) Show that the entire area of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ is $\frac{3}{8}\pi a^2$

$\boxed{OR}$ Find the area of the astroid. $x=a\cos^3t,\ y=a\sin^3t$



soln: (i) Since by putting $x=0, y=0$ both the sides of the curve are not satisfied, hence the curve does not pass through the origin.

(ii) After putting $x=-x$ and $y=-y$, given curve does not change, therefore the curve is symmetrical about both the axes.

(iii) Putting $y=0 \Rightarrow x^{2/3}=a^{2/3} \Rightarrow x^2=a^2 \Rightarrow x=\pm a$

i.e. the curve cuts the x-axis at $(a,0), (-a,0)$

Similarly $x=0 \Rightarrow y=\pm a$, the curve cuts the y-axis at $(0,a)$ and $(0,-a)$

Area of the curve = 4 × area of OAB

$\Rightarrow A = 4\int_0^a y\,dx$ — (1)

Parametric eqns $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$dx = -3a\cos^2\theta\sin\theta\, d\theta$

Also $x=0 \Rightarrow \theta=\pi/2$ and $x=a \Rightarrow \theta=0$

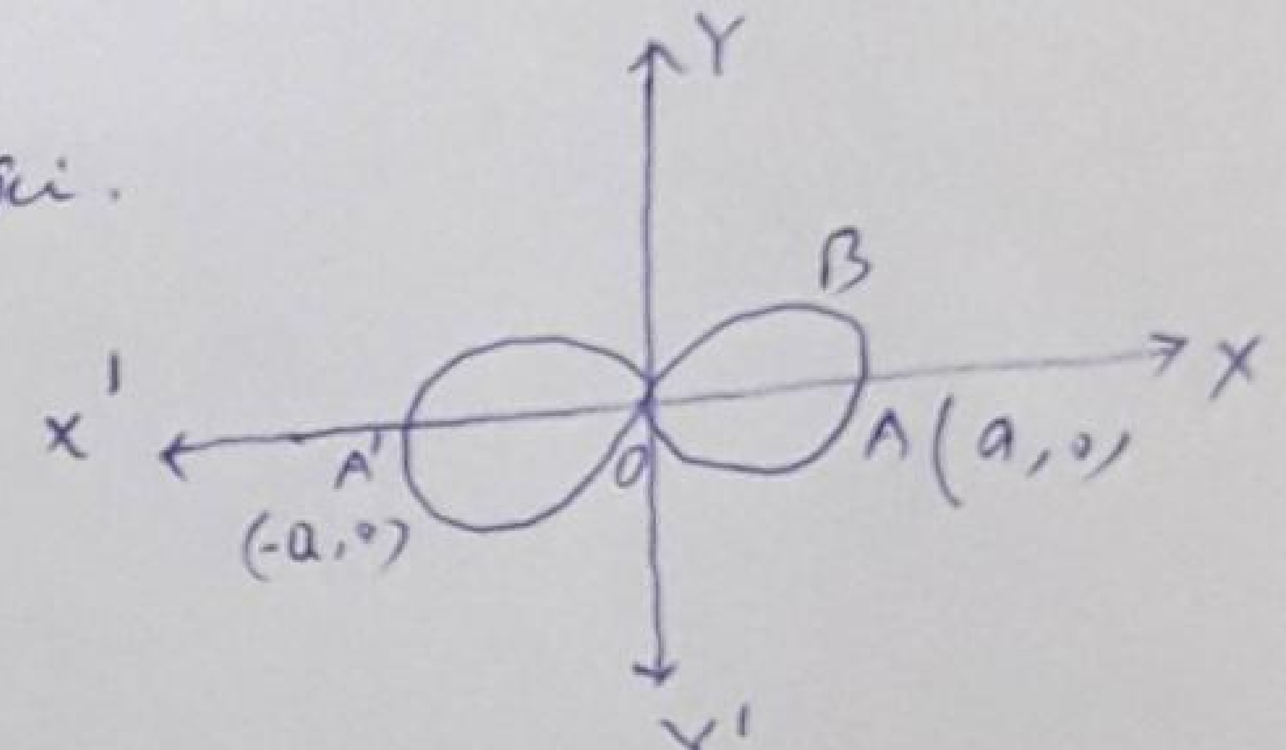
From ① $A = 4\int_{\pi/2}^{0} a\sin^3\theta\,(-3a\sin\theta\cos^2\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$ ②

$$A = 12a^2\int_0^{\pi/2}\sin^4\theta\,(1-\sin^2\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\,d\theta - 12a^2\int_0^{\pi/2}\sin^6\theta\,d\theta$$

$$= 12a^2\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{3\pi a^2}{8} \text{ Ans}$$

③ Find the whole area of the curve $a^2y^2 = x^2(a^2-x^2)$.

Sol:- (i) The curve passes through the origin.



(ii) The curve is symmetrical about both the axes.

(iii) It cuts the x-axis at $x=0, x=\pm a$.
It cuts the x-axis only at $y=0$.

(iv) There is no portion of the curve beyond $x=a$ or $x=-a$
y^2 is (−)ve and y is imaginary.

Area of the whole curve $= 4 \times$ area $OAB = 4\int_0^a y\,dx$

$$= 4\int_0^a \frac{x}{a}\sqrt{a^2-x^2}\,dx \quad \text{—— ①}$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$ Also $\left.\begin{array}{l} x=0 \Rightarrow \theta = 0 \\ x=a \Rightarrow \theta = \pi/2\end{array}\right\}$

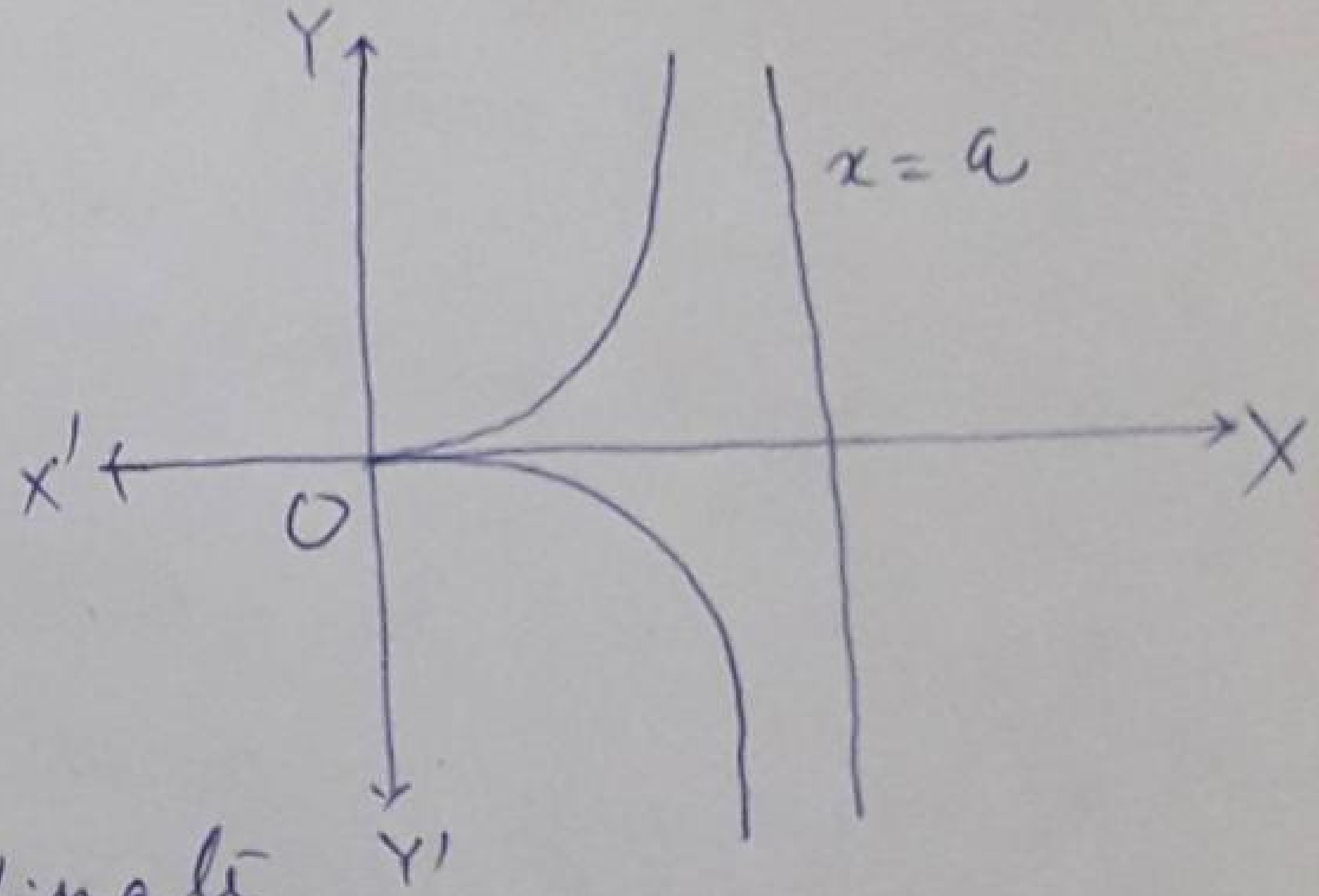
① becomes $= 4\int_0^{\pi/2}\sin\theta\cdot a\cos\theta\cdot a\cos\theta\,d\theta = 4a^2\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta$ —— ②

Let $\cos\theta = u \Rightarrow -\sin\theta\,d\theta = du$

From ② $= 4a^2\int_1^0 u^2(-du) = 4a^2\int_0^1 u^2\,du = \frac{4a^2}{3}$

(4) Obtain the area included between the curve $y^2(a-x) = x^3$ and its asymptotes.

Soln:-



(i) The curve passes through the origin, since both the sides are satisfied by putting $x=0, y=0$ in the eqn of the curve.

(ii) The curve is symmetrical about the x-axis.

(iii) The curve does not cut the co-ordinate axes at any other point except at the origin.

(iv) The x-axis is the tangent to the curve at the origin.

(v) The asymptote of the curve parallel to y-axis is $x=a$.

(vi) There is no portion of the curve on the L.H.S of $x=0$ or on the R.H.S of $x=a$.

$\therefore$ Required area = 2 × area of the upper half

$$= 2\int_0^a y\,dx = 2\int_0^a \frac{x\sqrt{x}}{\sqrt{a-x}}\,dx \quad \text{—— (1)}$$

To evaluate (1), put $x = a\sin^2\theta$

$\Rightarrow dx = 2a\sin\theta\cos\theta\,d\theta$

when $x=0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \pi/2$.

(1) becomes

$$= 2\int_0^{\pi/2} \frac{a\sin^2\theta\cdot\sqrt{a}\sin\theta}{\sqrt{a}\cos\theta}\cdot 2a\sin\theta\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2}\sin^4\theta\,d\theta = 4a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{4} \text{ Ans}$$

(5) Find the area of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ from ~~to~~ cusp to cusp.

Soln:- From the curve $y=0$ when $\theta = 0$ to 2π.

$x = a(\theta - \sin\theta) \Rightarrow dx = a(1-\cos\theta)\,d\theta$

Required area $= \int y\,dx = \int_0^{2\pi} a(1-\cos\theta)\cdot a(1-\cos\theta)\,d\theta = a^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$

$$= a^2\int_0^{2\pi}(1+\cos^2\theta - 2\cos\theta)\,d\theta = a^2\int_0^{2\pi}\left(1+\frac{1+\cos 2\theta}{2} - 2\cos\theta\right)d\theta$$

$$= \frac{a^2}{2}\int(3 - 4\cos\theta + \cos 2\theta)\,d\theta = 3\pi a^2$$

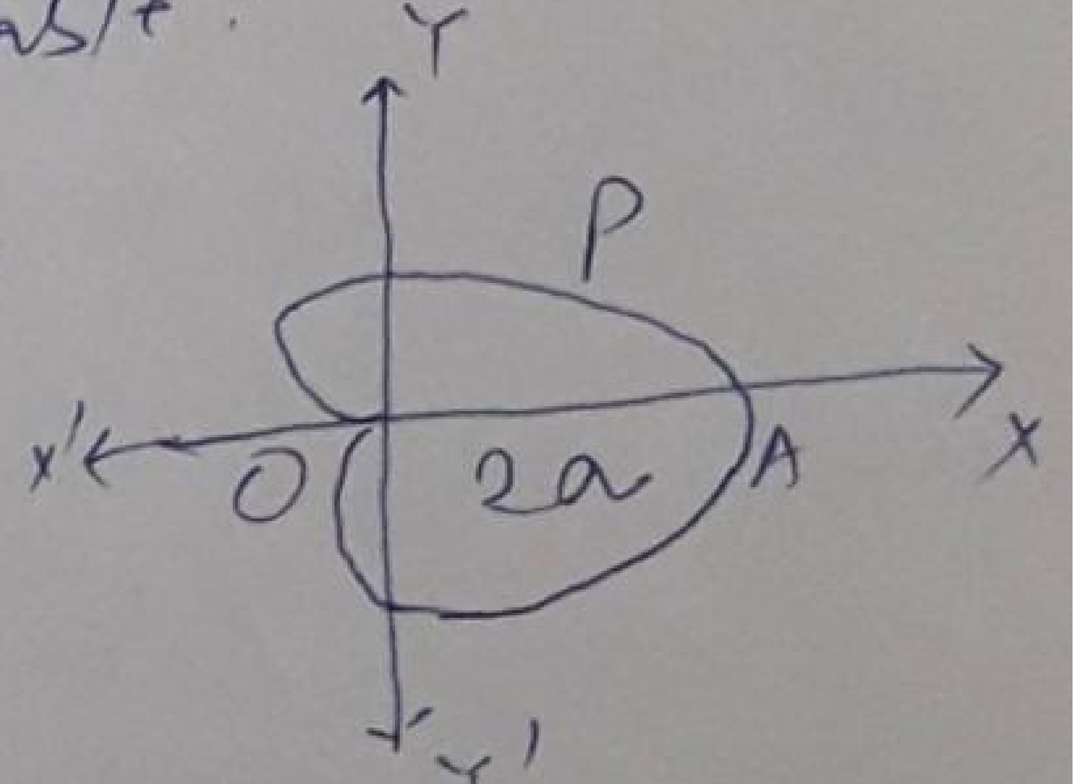
⑥ Find the area of the Cardiode $r = a(1+\cos\theta)$

Sol:: (i) Replacing $-\theta$ for θ in the eqn of the given curve, the eqn does not change. Therefore the curve is symmetrical about the initial line.

(ii) $r = 0 \Rightarrow \theta = \pi$; $\theta = 0 \Rightarrow r = 2a$.

θ	0	30°	45°	60°	90°	...	180°
r	$2a$	$a\left(1+\frac{\sqrt3}{2}\right)$	$a\left(1+\frac{1}{\sqrt2}\right)$	$\frac{3a}{2}$	a	...	0

As θ increases from 0 to π, r decreases from $2a$ to a as is evident from the corresponding values of r and θ given by the following table.



∴ Required area $= 2 \times$ area $OAP = 2 \times \frac{1}{2}\int_0^{\pi} r^2\, d\theta$

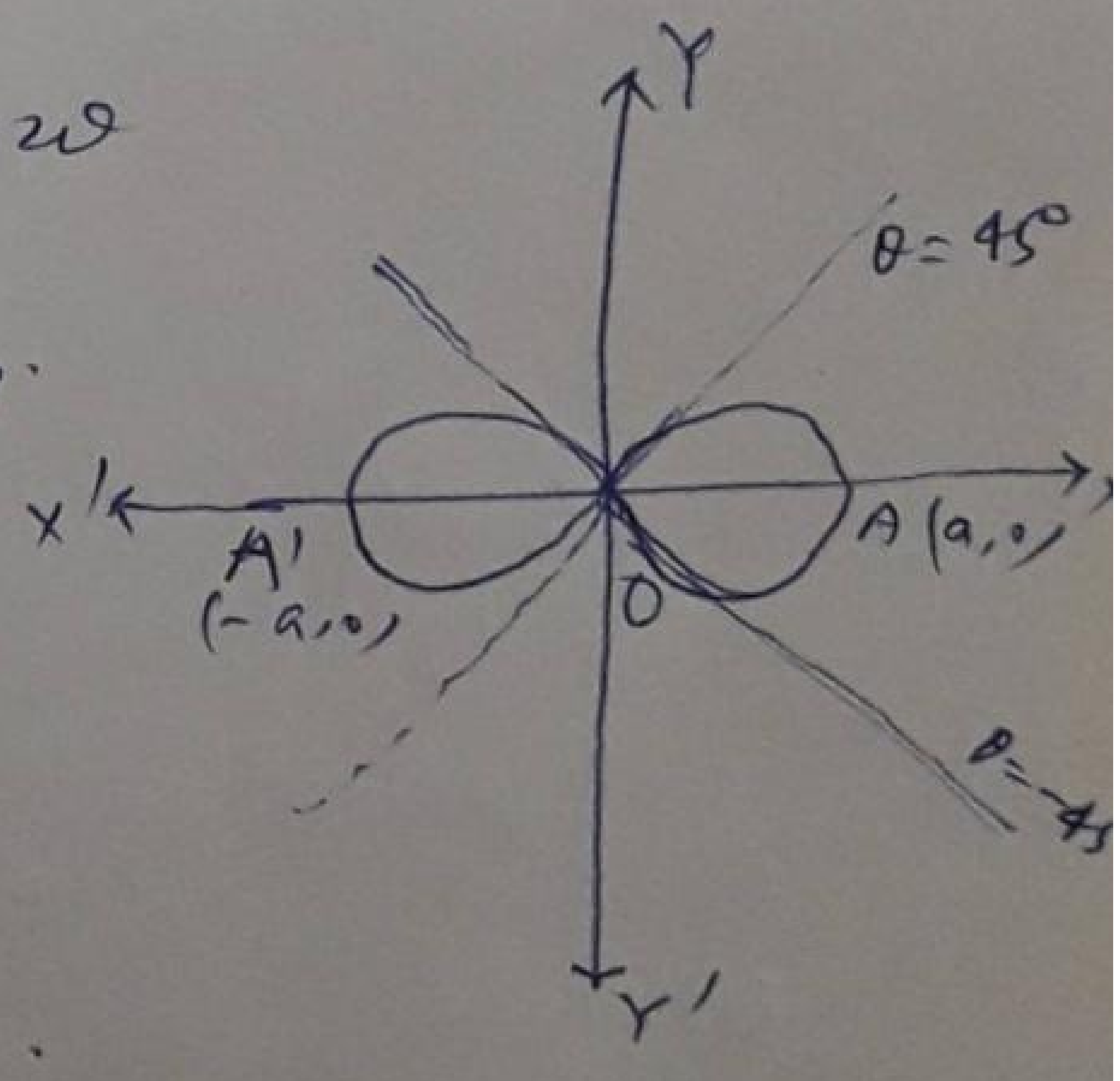
$$= \int_0^{\pi} a^2(1+\cos\theta)^2 d\theta = a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\left[\pi + 2\{\sin\theta\}_0^{\pi} + 2\int_0^{\pi/2}\cos^2\theta\, d\theta\right]$$

$$= a^2\left[\pi + 0 + 2\times\frac{1}{2}\times\frac{\pi}{2}\right] = \frac{3}{2}\pi a^2$$

⑦ Find the area of the loop of the curve $r^2 = a^2\cos 2\theta$

Sol:: (i) Replacing $-\theta$ for θ, curve does not change i.e. curve is symmetrical about the initial line.



(ii) $r = 0 \Rightarrow \theta = \pm\pi/4$ and $\theta = 0 \Rightarrow r = \pm a$.

$\theta = 30^\circ$, $r = \pm a/\sqrt2$, $\theta = 45^\circ$, $r = 0$

$\theta = 60^\circ$, $r^2 = -$ve. , $\theta = 90^\circ$, $r^2 = -$ve.

Put $r\cos\theta = x$ and $r\sin\theta = y$ in $r^2 = a^2\cos 2\theta$.

$\Rightarrow r^2 = a^2(\cos^2\theta - \sin^2\theta) \Rightarrow (x^2+y^2)^2 = a^2(x^2-y^2)$

Equating the lowest degree term to zero we get $x^2 - y^2 = 0 \Rightarrow y = \pm x$. therefore the tangents to the curve at the origin are $y = \pm x$.

Area of one loop $= 2 \times$ area $OAP = 2 \times \frac{1}{2}\int_0^{\pi/4} r^2\, d\theta = \int_0^{\pi/4} a^2\cos 2\theta\, d\theta$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{a^2}{2}.$$

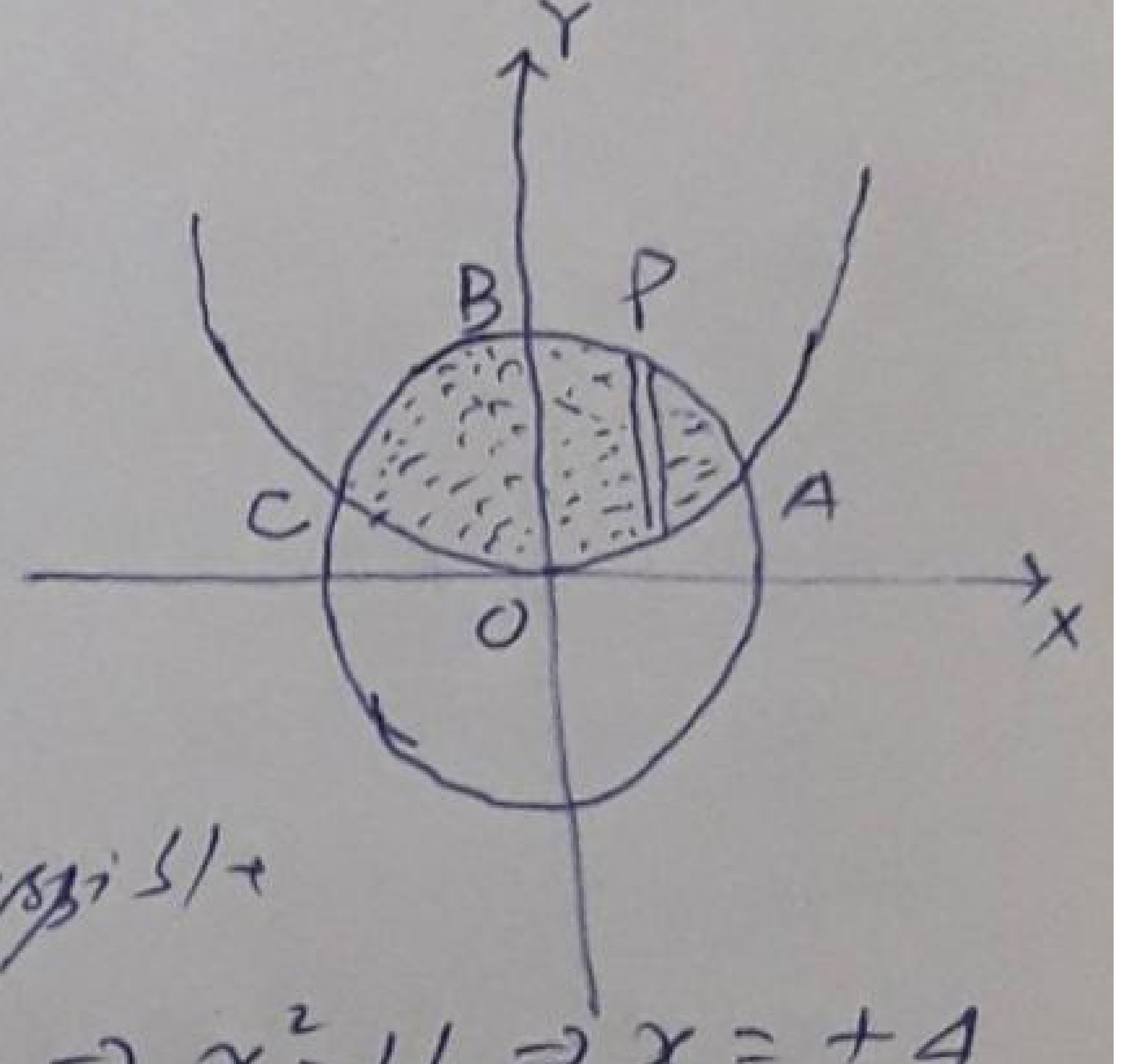
Required area = area of the two equal loops

$$= 2\times\frac{a^2}{2} = a^2$$

⑧ Find the area common to the circle $x^2+y^2=25$ and the parabola $3x^2=16y$. ⑤

Sol:- $x^2+y^2=25$ represents a circle with centre $(0,0)$, radius $=5$ $\boxed{x^2+y^2=a^2}$

and $3x^2=16y$ represents a parabola $\boxed{x^2=4ay}$ whose vertex is the origin and the axis is the y-axis.



solving $x^2+y^2=25$ and $3x^2=16y$ we get the points of intersection of the two curves.

$$\frac{16}{3}y+y^2=25 \Rightarrow 3y^2+16y-75=0$$
$$\Rightarrow (y-3)(3y+25)=0$$

$\therefore y=3, -\frac{25}{3}$ (-)ve value is inadmissible

$\therefore y=3$ and $x^2+y^2=25 \Rightarrow x^2+9=25 \Rightarrow x^2=16 \Rightarrow x=\pm 4$

Thus the Co-ordinates of $A=(4,3)$.

Now the area common to the two curves $= 2\times$ area AOB

$$= 2\int_0^4 (y_1-y_2)\,dx$$

where y_1 is taken for the circle $x^2+y^2=25$ and y_2 is taken for the parabola $3x^2=16y$

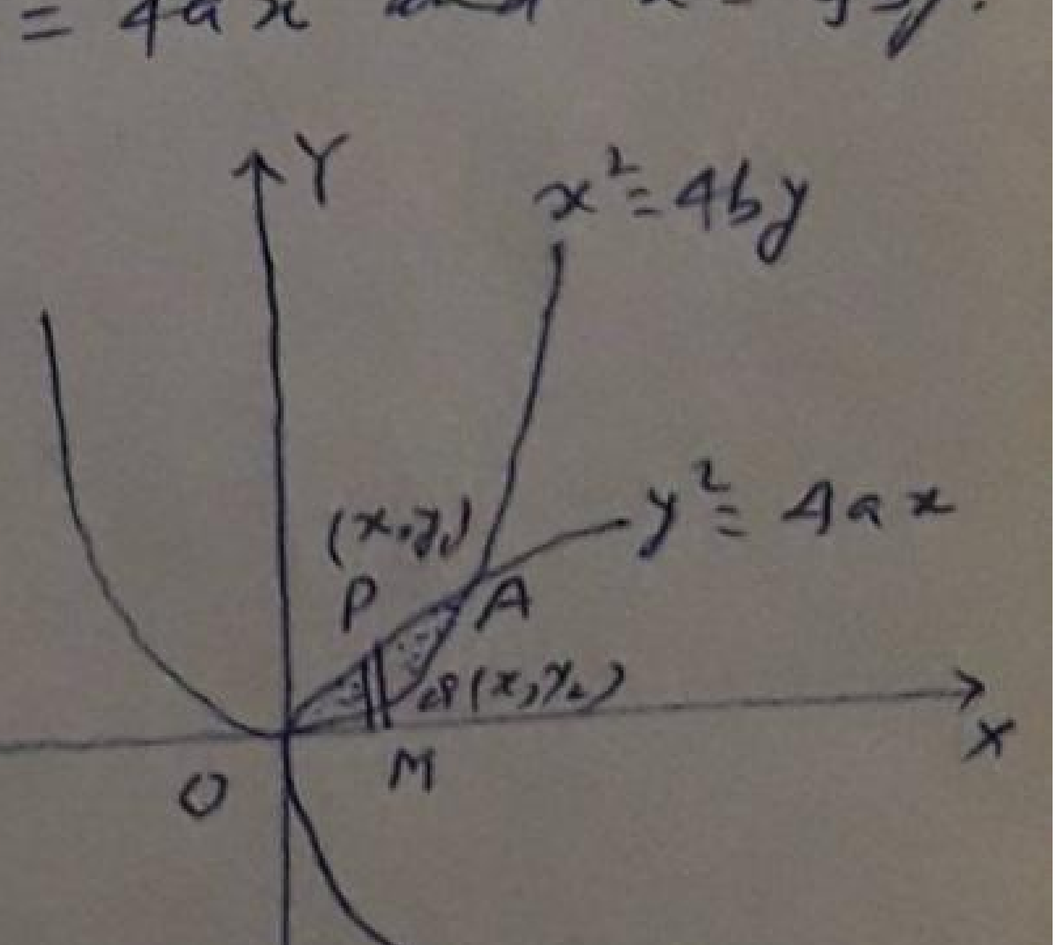
$$= 2\times\int_0^4\left(\sqrt{25-x^2}-\frac{3}{16}x^2\right)dx$$

$$= 2\times\left[\left\{\frac{x\sqrt{25-x^2}}{2}+\frac{25}{2}\sin^{-1}\frac{x}{5}\right\}-\frac{3}{16}\cdot\frac{x^3}{3}\right]_0^4$$

$$= 2\left[6+\frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right)-4\right] = 4+25\sin^{-1}(4/5)\ \Omega$$

⑨ Find the area enclosed by the parabola $y^2=4ax$ and $x^2=4by$.

Sol:: $y^2=4ax$ — ① and $x^2=4by$ — ②



$\Rightarrow y^4=16a^2x^2$

$= 16a^2\cdot 4by$ {from ②}

$\Rightarrow y(y^3-64a^2b)=0 \Rightarrow y=0$ & $4a^{2/3}b^{1/3}$

Now, $y=0 \Rightarrow x=0$ and from ① $y=4a^{2/3}b^{1/3}$
$\Rightarrow x=4a^{1/3}b^{2/3}$ }

$\therefore$ Points of intersection are $O(0,0)$ and $A\left(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3}\right)$

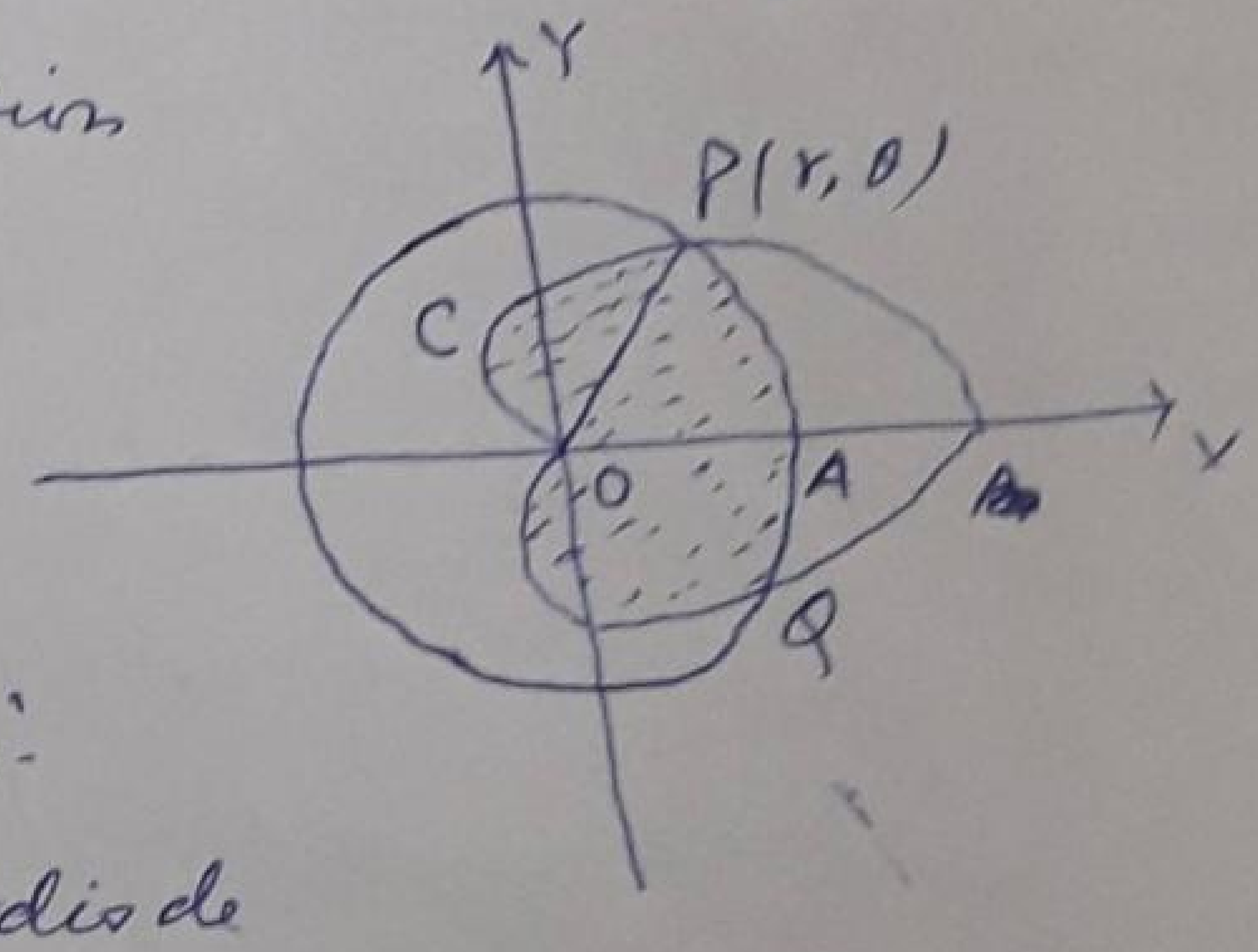
Therefore the area enclosed between the two parabola is

$$= \int_0^{4a^{1/3}b^{2/3}} (y_1 - y_2)\,dx = \int_0^{4a^{1/3}b^{2/3}} \left(\sqrt{4ax} - \frac{x^2}{4b}\right) dx$$

Since y_1 lies on $y^2 = 4ax$ and y_2 lies on $x^2 = 4by$

$$= \left[2\sqrt{a}\cdot\frac{2}{3}x^{2/3} - \frac{1}{12}x^3\right]_0^{4a^{1/3}b^{2/3}} = \left[\frac{32}{3}ab - \frac{16}{3}ab\right] = \frac{16}{3}ab$$

(12) Find the area common to the circle $r = \frac{3}{2}a$ and the cardiode $r = a(1+\cos\theta)$ and also the area of the remainder of the cardiode



Sol:- Let $P(r,\theta)$ be the point of intersection of the circle and the cardiode.

Now $r = a(1+\cos\theta)$ and $r = \frac{3}{2}a$

$\Rightarrow \frac{3}{2}\cancel{a} = \cancel{a}(1+\cos\theta) \Rightarrow 1+\cos\theta = \frac{3}{2}$

$\cos\theta = \frac{3}{2} - 1 = \frac{1}{2} = \cos 60^\circ \Rightarrow \theta = 60^\circ$

The area common to the circle and the cardiode

$= 2\times \text{area } OAPCO = 2(\text{area } OAP + \text{area } OPCO)$

Area $OAP = \frac{1}{2}\int_0^{\pi/3} r^2\,d\theta$ where $r = \frac{3}{2}a$

$$= \frac{1}{2}\int_0^{\pi/3} \frac{9}{4}a^2\,d\theta = \frac{9a^2}{8}\int_0^{\pi/3} d\theta = \frac{\cancel{9}^{3}a^2}{8}\cdot\frac{\pi}{\cancel{3}} = \frac{3\pi a^2}{8}$$

Area $OPCO = \frac{1}{2}\int_{\pi/3}^{\pi} r^2\,d\theta$, where $r = a(1+\cos\theta)$

$$= \frac{1}{2}\int_{\pi/3}^{\pi} a^2(1+\cos\theta)^2\,d\theta = \frac{a^2}{2}\int_{\pi/3}^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta = \frac{a^2}{2}\int_{\pi/3}^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{4}\int_{\pi/3}^{\pi}(2+4\cos\theta+1+\cos 2\theta)\,d\theta = \frac{a^2}{4}\left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi}$$

$$= \frac{a^2}{4}\left[\left\{3\pi + 4\sin\pi + \frac{1}{2}\sin 2\pi\right\} - \left\{3\left(\frac{\pi}{3}\right) + 4\sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right\}\right]$$

$$= \frac{a^2}{4}\left(2\pi - \frac{9\sqrt{3}}{4}\right)$$

$\therefore$ Required area $= 2\left\{\frac{3\pi a^2}{8} + \frac{a^2}{4}\left(2\pi - \frac{9\sqrt{3}}{4}\right)\right\} = \left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8}\right)a^2$

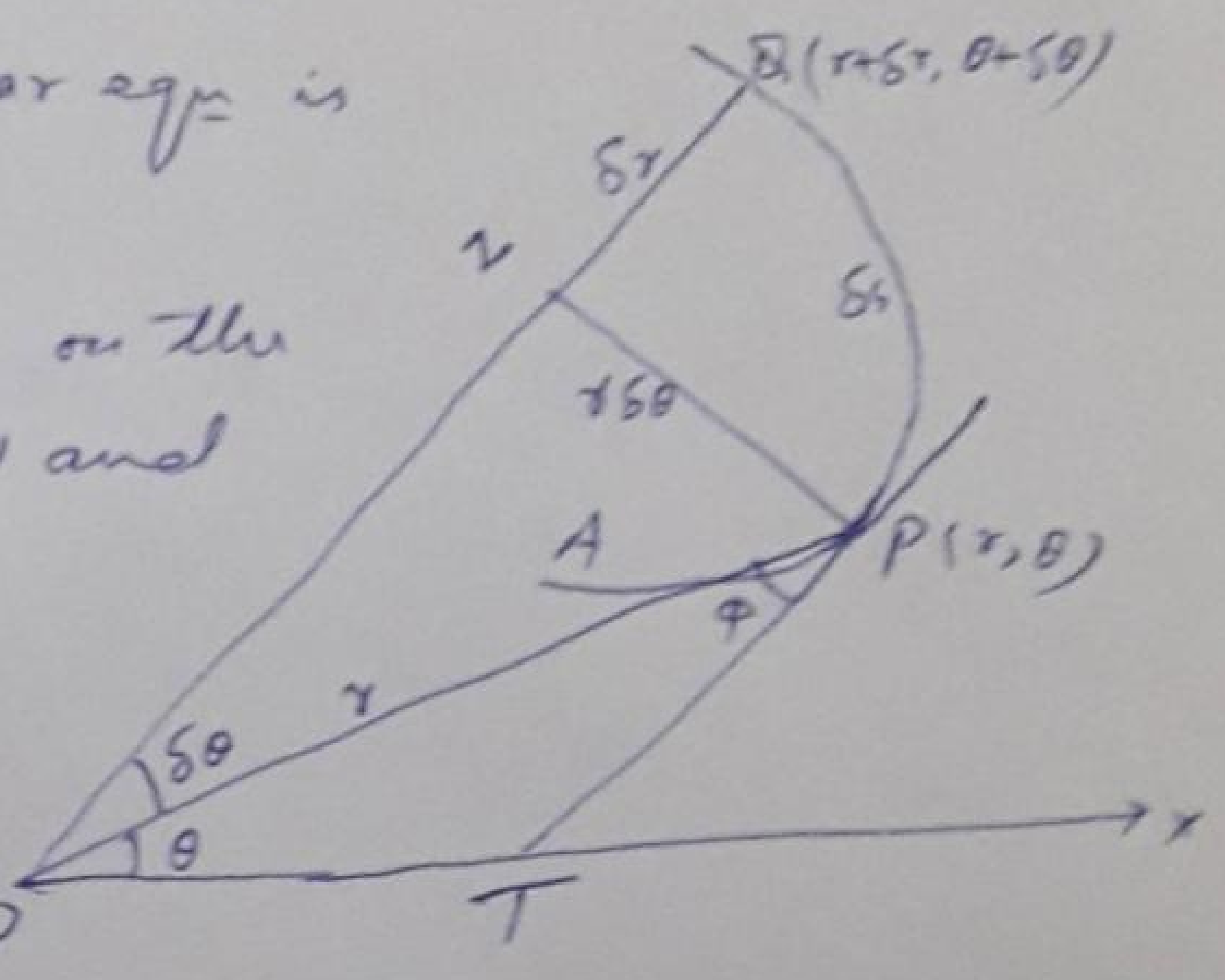
Area of the remainder cardiode $= \frac{3}{2}\pi a^2 - \left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{4}\right)a^2 = \left(\frac{9\sqrt{3}}{8} - \frac{\pi}{4}\right)a^2$

= (whole area of card − area common to circle & cardiode) −

Lengths Determined from Polar Equation

(Art) Let AB be a curve whose polar eqn is

$$r = f(\theta)$$



Let P and Q be two consecutive points on the curve whose co-ordinates are (r, θ) and $(r+\delta r, \theta+\delta\theta)$ respectively so that

$\angle POQ = \delta\theta$ and $OQ = r + \delta r$

Let $AP = s$, then

$AQ = s + \delta s \Rightarrow PQ = \delta s$

Draw $PN \perp OQ$. Then $PN = r\sin\delta\theta$ and $ON = r\cos\delta\theta$.

In the limit when $Q \to P$, $\delta\theta \to 0$ so that PN becomes $= r\delta\theta$ and $ON = r$, consequently $NQ = \delta r$.

Also $\dfrac{\text{arc } PQ}{\text{Chord } PQ} \to 1$

In rt. angled $\triangle PNQ$,

$PQ^2 = PN^2 + NQ^2$

$(\delta s)^2 = (r\delta\theta)^2 + (\delta r)^2$

$\Rightarrow \left(\dfrac{\delta s}{\delta\theta}\right)^2 = r^2 + \left(\dfrac{\delta r}{\delta\theta}\right)^2$

Taking limit $\delta\theta \to 0$, $\therefore \left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2$

$\therefore \dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} \Rightarrow \int ds = \int \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\, d\theta$.

$$\boxed{s = \int_A^B \sqrt{r^2 + (dr/d\theta)^2}\, d\theta}$$

Let the angle between the radius vector OP and the tangent PT at P be ϕ. Then from differential Calculus, $\tan\phi = r\dfrac{d\theta}{dr}$.

$\sin\phi = \dfrac{r\,d\theta}{ds}$, $\cos\phi = \dfrac{dr}{ds}$.

Again, $(\delta s)^2 = (\delta r)^2 + (r\delta\theta)^2$

$\left(\dfrac{ds}{dr}\right)^2 = 1 + r^2\left(\dfrac{d\theta}{dr}\right)^2$ — As lim $\delta r \to 0$

$\Rightarrow \int ds = \int \sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}\, dr$

$$\therefore \boxed{s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}\, dr}$$

(Art) Lengths Determined from Pedal Equation

Let the Pedal eqn of the curve be $p = f(r)$

Then from diff. Calculus, $\frac{dr}{ds} = \cos\phi$, $r\frac{d\theta}{dr} = \tan\phi$

and ~~where~~ $p = r\sin\phi$ where p = length of the $\perp$ from the pole to the tangent.

Now $(ds)^2 = (dr)^2 + (r\,d\theta)^2$

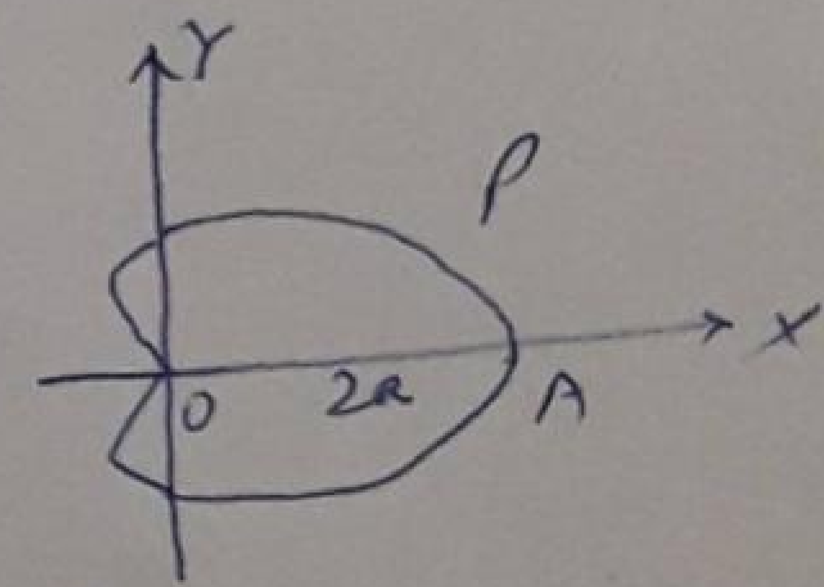
$$\Rightarrow \left(\frac{ds}{dr}\right)^2 = 1 + \left(r\frac{d\theta}{dr}\right)^2 = 1 + \tan^2\phi = \sec^2\phi = \frac{1}{\cos^2\phi}$$

$$= \frac{1}{1-\sin^2\phi} = \frac{1}{1-\frac{p^2}{r^2}} = \frac{r^2}{r^2-p^2}.$$

$$\frac{ds}{dr} = \frac{r}{\sqrt{r^2-p^2}} \Rightarrow \int ds = \int \frac{r\,dr}{\sqrt{r^2-p^2}}$$

$$\therefore \boxed{s = \int_{r_1}^{r_2} \frac{r\,dr}{\sqrt{r^2-p^2}}}$$

Ex-1 Find the entire length of the Cardiode $r = a(1+\cos\theta)$.



Sol:- $r = a(1+\cos\theta)$

$$\frac{dr}{d\theta} = -a\sin\theta$$

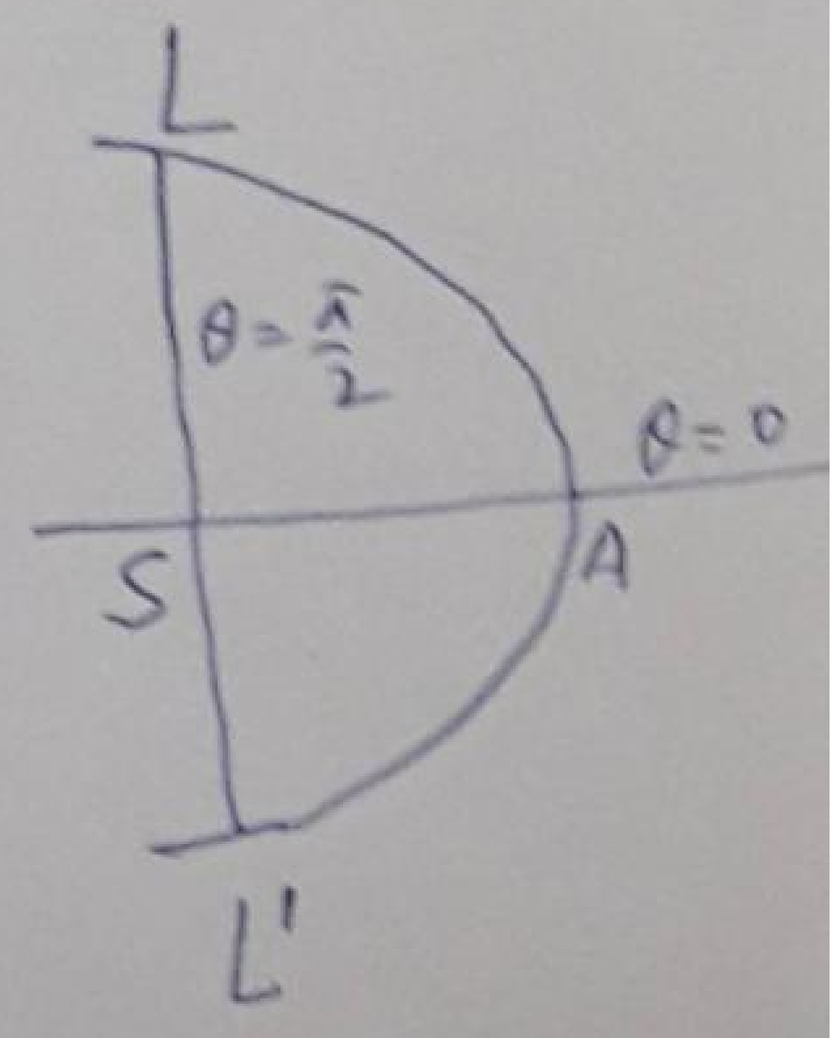
$$\therefore \text{whole length} = 2\int_0^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta = 2a\int_0^{\pi}\sqrt{2+2\cos\theta}\,d\theta$$

$$= 2a\int_0^{\pi}\sqrt{2\times 2\cos^2\frac{\theta}{2}}\,d\theta = 4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 4a\cdot 2\left[\sin\frac{\theta}{2}\right]_0^{\pi}$$

$$= 8a$$

Ex-2 Find the length of the parabola $\frac{2a}{r} = 1+\cos\theta$ cut off by the latus rectum.



Sol^n:- focus S is the pole

$$\therefore \frac{2a}{r} = 1+\cos\theta$$

$$r = \frac{2a}{1+\cos\theta} = \frac{\cancel{2}a}{\cancel{2}\cos^2\theta/2} = a\sec^2\frac{\theta}{2}.$$

So that $$\frac{dr}{d\theta} = a.\cancel{2}\sec\frac{\theta}{2}.\sec\frac{\theta}{2}.\tan\frac{\theta}{2}.\cancel{\frac{1}{2}}$$

$$= a\sec^2\frac{\theta}{2}.\tan\frac{\theta}{2}$$

The length of the arc $LAL' = 2\times \text{arc } AL = 2\int_0^{\pi/2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$= 2\int_0^{\pi/2}\sqrt{a^2\sec^4\frac{\theta}{2}+a^2\sec^4\frac{\theta}{2}\tan^2\frac{\theta}{2}}\;d\theta$$

$$= 2\int_0^{\pi/2}\sqrt{a^2\sec^4\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)}\,d\theta = 2a\int_0^{\pi/2}\sec^2\frac{\theta}{2}\sqrt{1+\tan^2\frac{\theta}{2}}\,d\theta \quad \text{——(1)}$$

Let $\tan\frac{\theta}{2} = u$ so that $\sec^2\frac{\theta}{2}.\frac{d\theta}{2} = du$

Also $\theta = 0 \Rightarrow u = 0$ and $\theta = \frac{\pi}{2} \Rightarrow u = \tan\frac{\pi}{4} = 1$

$\therefore$ The required length of arc, from (1) $= 2a\int_0^1\sqrt{1+u^2}.\,2du$

$$= 4a\left[\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\log\left\{u+\sqrt{1+u^2}\right\}\right]_0^1$$

$$= 4a\left[\frac{1}{2}\sqrt{2}+\frac{1}{2}\log(1+\sqrt{2})\right] = 2a\left\{\sqrt{2}+\log(1+\sqrt{2})\right\}.$$

Ex-3 Find the intrinsic eq^n of the curve $r = a(1-\cos\theta)$

Sol^n:- $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\frac{1}{r}.\frac{dr}{d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)} = \frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^2\theta/2}$$

$$\frac{1}{\tan\phi} = \cot\frac{\theta}{2} \Rightarrow \cot\phi = \cot\frac{\theta}{2}$$

$$\therefore \phi = \frac{\theta}{2}$$

working rule

(i) $s = \int_\alpha^\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

(ii) $\psi = \theta+\phi$

(iii) $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta}$

Consequently $\psi = \theta + \phi = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$ —— (1) (10)

Also $s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= \int_0^\theta \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta = a\int_0^\theta \sqrt{2(1-\cos\theta)}\, d\theta$

$= \sqrt{2}\, a \int_0^\theta \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta = 2a\int_0^\theta \sin\frac{\theta}{2}\, d\theta = \cancel{2a\left[-2\cos\theta\right.}$

$= 2a\left[-2\cos\frac{\theta}{2}\right]_0^\theta = 4a\left(1-\cos\frac{\theta}{2}\right)$ —— (2)

Eliminating θ between (1) & (2) we get

$s = 4a\left(1-\cos\frac{\psi}{3}\right)$ which is the intrinsic eqn of the curve.

— x —