Class- Degree - I (H+S) MATHS

TRIGONOMETRY

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Ex Obtain the roots of $x^4 - x^3 + x^2 - x + 1 = 0$

Sol^n: we have, $x^4 - x^3 + x^2 - x + 1 = 0$

$\Rightarrow (x+1)(x^4 - x^3 + x^2 - x + 1) = 0$

$\Rightarrow x^5 + 1 = 0 \Rightarrow x^5 = -1$

$\Rightarrow x^5 = \cos(2r\pi + \pi) + i\sin(2r\pi + \pi)$

$\Rightarrow x = \cos(2r+1)\frac{\pi}{5} + i\sin(2r+1)\frac{\pi}{5}$ where $r = 0, 1, 2, 3, 4$.

$r = 0$, $x_1 = \cos\pi/5 + i\sin\pi/5$

$r = 1$, $x_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$r = 2$, $x_3 = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5} = -1$

$r = 3$, $x_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} = \cos(2\pi - \frac{3\pi}{5}) + i\sin(2\pi - \frac{3\pi}{5}) = \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

$r = 4$, $x_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} = \cos(2\pi - \frac{\pi}{5}) + i\sin(2\pi - \frac{\pi}{5}) = \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

$\therefore$ The roots are $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$, -1

Ex solve $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

Sol^n:- $(x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0$

$\Rightarrow x^7 - 1 = 0 \Rightarrow x^7 = 1 = \cos(2r\pi) + i\sin(2r\pi)$

$\therefore x = \cos\frac{2r\pi}{7} + i\sin\frac{2r\pi}{7}$. $r = 0, 1, 2, 3, 4, 5, 6$.

$r = 0$, $x_1 = \cos 0 + i\sin 0 = 1$.

$r = 1$, $x_2 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ etc.

Ex If α, β, γ be real angles such that
$\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$,
prove that $\cos n\alpha + \cos n\beta + \cos n\gamma = 3\cos\frac{n}{3}(\alpha+\beta+\gamma)$ or 0
according as n is or not a multiple of 3.

Sol^n:- $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$

$\Rightarrow 2(\cos\beta\cdot\cos\gamma + \sin\beta\cdot\sin\gamma) + 2(\cos\gamma\cdot\cos\alpha + \sin\gamma\cdot\sin\alpha) +$
$(\cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta) = -1-1-1 = -(\cos^2\alpha + \sin^2\alpha) - (\cos^2\beta + \sin^2\beta) - (\cos^2\gamma + \sin^2\gamma)$.

$\Rightarrow (\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cdot\cos\beta + 2\cos\beta\cdot\cos\gamma + 2\cos\gamma\cdot\cos\alpha) +$
$(\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) = 0$

$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0$ and
$\sin\alpha + \sin\beta + \sin\gamma = 0$

Let $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$

$\therefore x + y + z = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

Also, $\frac{1}{x} = x^{-1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$\frac{1}{y} = y^{-1} = (\cos\beta + i\sin\beta)^{-1} = \cos\beta - i\sin\beta$

$\frac{1}{z} = z^{-1} = (\cos\gamma + i\sin\gamma)^{-1} = \cos\gamma - i\sin\gamma$

$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$\Rightarrow \frac{xy + yz + zx}{xyz} = 0$

$\therefore xy + yz + zx = 0$

Now, $(1+ax)(1+ay)(1+az)$

$= 1 + a(x+y+z) + a^2(xy + yz + zx) + a^3xyz$

$= 1 + a\cdot 0 + a^2\cdot 0 + a^3xyz$

Taking logarithm on both sides.

$\log(1+ax) + \log(1+ay) + \log(1+az) = \log(1 + a^3xyz)$

$ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \dots + (-1)^{n-1}\frac{(ax)^n}{n} + ay - \frac{(ay)^2}{2} + \frac{(ay)^3}{3} - \dots + (-1)^{n-1}\frac{(ay)^n}{n} + az - \frac{(az)^2}{2} + \frac{(az)^3}{3} - \dots + (-1)^{n-1}\frac{(az)^n}{n}$

$= a^3xyz - \frac{(a^3xyz)^2}{2} + \frac{(a^3xyz)^3}{3} - \dots + (-1)^{n-1}\frac{(a^3xyz)^n}{n} + \dots$

Equating the co-efficient of a^n

$x^n + y^n + z^n = 0$ when n is not a multiple of 3

$\Rightarrow (\cos\alpha + i\sin\alpha)^n + (\cos\beta + i\sin\beta)^n + (\cos\gamma + i\sin\gamma)^n = 0$

$\Rightarrow (\cos n\alpha + \cos n\beta + \cos n\gamma) + i(\sin n\alpha + \sin n\beta + \sin n\gamma) = 0$

Equating real part: $\cos n\alpha + \cos n\beta + \cos n\gamma = 0$

Also $x^n + y^n + z^n = 3(xyz)^{\frac{n}{3}}$, when n is a multiple of 3

$\Rightarrow (\cos\alpha + i\sin\alpha)^n + (\cos\beta + i\sin\beta)^n + (\cos\gamma + i\sin\gamma)^n = 3\left[(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\right]^{n/3}$

$\Rightarrow \cos n\alpha + \cos n\beta + \cos n\gamma = 3\cos\frac{n}{3}(\alpha + \beta + \gamma)$

(Equating real parts only)

① Prove that $\theta \cot\theta = 1 - \frac{1}{3}\theta^2 - \frac{1}{45}\theta^4$, when θ is small.

Sol^n: $\theta\cot\theta = \theta\frac{\cos\theta}{\sin\theta} = \dfrac{\theta\left(1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\cdots\right)}{\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\cdots}$

$= \left[1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\cdots\right]\left[1-\left\{\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\cdots\right\}\right]^{-1}$

$(1+x)^n = 1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\cdots$
$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{\lfloor 2}x^2+\cdots$

$= \left(1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\cdots\right)\left[1+\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\cdots\right)+\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\cdots\right)^2+\cdots\right]$

$= 1+\left(\frac{1}{\lfloor 3}-\frac{1}{\lfloor 2}\right)\theta^2+\left(\frac{1}{\lfloor 4}-\frac{1}{\lfloor 3\lfloor 2}-\frac{1}{\lfloor 5}+\frac{1}{(\lfloor 3)^2}\right)\theta^4+\cdots$

[neglecting higher powers of θ]

$= 1-\frac{\theta^2}{3}-\frac{1}{45}\theta^4.$

② Prove that $\sin^2\theta\cos\theta = \theta^2-\frac{5}{6}\theta^4+\cdots+(-1)^{n+1}\frac{3^{2n}-1}{4\lfloor 2n}\theta^{2n}+\cdots$

Sol^n: $\sin^2\theta\cdot\cos\theta = \frac{1}{2}(2\sin\theta\cos\theta)\sin\theta = \frac{1}{2}(\sin 2\theta\cdot\sin\theta) = \frac{1}{4}(\cos\theta-\cos 3\theta)$

$= \frac{1}{4}\left[\left(1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\cdots+(-1)^n\frac{\theta^{2n}}{\lfloor 2n}+\cdots\right)-\left(1-\frac{3^2\theta^2}{\lfloor 2}+\frac{3^4\theta^4}{\lfloor 4}-\cdots+(-1)^n\frac{3^{2n}\theta^{2n}}{\lfloor 2n}-\cdots\right)\right]$

$= \frac{\theta^2}{4}\left(\frac{3^2}{\lfloor 2}-\frac{1}{\lfloor 2}\right)-\frac{\theta^4}{4}\left(-\frac{1}{\lfloor 4}+\frac{3^4}{\lfloor 4}\right)+\cdots+(-1)^{n+1}\frac{3^{2n}-1}{4\lfloor 2n}\theta^{2n}+\cdots$

$= \theta^2-\frac{5}{6}\theta^4+\cdots+(-1)^{n+1}\frac{3^{2n}-1}{4\lfloor 2n}\theta^{2n}+\cdots$

③ Prove that $\frac{1}{6}\sin^3\theta = \frac{\theta^3}{\lfloor 3}-(1+3^2)\frac{\theta^5}{\lfloor 5}+(1+3^2+3^4)\frac{\theta^7}{\lfloor 7}-\cdots$

Sol^n: $\because \sin^3\theta = \frac{1}{4}(3\sin\theta-\sin 3\theta)$

$\sin 3A = 3\sin A-4\sin^3 A$
$\Rightarrow \sin^3 A = \frac{3\sin A-\sin 3A}{4}$

$= \frac{1}{4}\left[3\left(\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\frac{\theta^7}{\lfloor 7}+\cdots\right)-\left(3\theta-\frac{3^3\theta^3}{\lfloor 3}+\frac{3^5\theta^5}{\lfloor 5}-\frac{3^7\theta^7}{\lfloor 7}+\cdots\right)\right]$

$= \frac{\theta^3}{4}\left(\frac{3^3}{\lfloor 3}-\frac{3}{\lfloor 3}\right)+\frac{\theta^5}{4}\left(\frac{3}{\lfloor 5}-\frac{3^5}{\lfloor 5}\right)+\frac{\theta^7}{4}\left(\frac{3^7}{\lfloor 7}-\frac{3}{\lfloor 7}\right)+\cdots$

$= \frac{\theta^3}{\lfloor 3}\cdot\frac{3^3-3}{4}-\frac{\theta^5}{\lfloor 5}\cdot\frac{3^5-3}{4}+\frac{\theta^7}{\lfloor 7}\cdot\frac{3^7-3}{\lfloor 4}-\cdots$

$= \frac{\theta^3}{\lfloor 3}\cdot\frac{3(3^2-1)}{4}-\frac{\theta^5}{\lfloor 5}\cdot\frac{3(3^4-1)}{4}+\frac{\theta^7}{\lfloor 7}\cdot\frac{3(3^6-1)}{4}-\cdots$

$$\frac{1}{6}\sin^3\theta = \frac{\theta^3}{\lfloor 3} - \frac{\theta^5}{\lfloor 5}\cdot\frac{3(3^2+1)(3^2-1)}{6\cdot 4} + \frac{\theta^7}{\lfloor 7}\cdot\frac{3(3^4+3^2+1)(3^2-1)}{4\cdot 6} - \cdots \quad (4)$$

$$= \frac{\theta^3}{\lfloor 3} - (1+3^2)\frac{\theta^5}{\lfloor 5} + (1+3^2+3^4)\frac{\theta^7}{\lfloor 7} - \cdots$$

(4) Prove that $1+\cos 10\theta = 2(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)^2$

Sol^n: $1+\cos 10\theta = 1+\cos 2(5\theta) = 2\cos^2 5\theta = 2(\cos 5\theta)^2$

$$= 2\left[\cos^5\theta - \frac{5(5-1)}{\lfloor 2}\cos^3\theta\cdot\sin^2\theta + \frac{5(5-1)(5-2)(5-3)}{\lfloor 4}\cos\theta\sin^4\theta\right]^2$$

$$\boxed{\because \cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \cdots}$$

$$= 2\left[\cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2\right]^2$$

$$= 2\left[\cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta\right]^2$$

$$= 2\left[16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right]^2$$

(5) Expand $\frac{1-\cos x}{\sin x}$ in ascending powers of x upto three places.

Sol^n:
$$\frac{1-\cos x}{\sin x} = \frac{\cancel{2}\sin^{\cancel{2}}(x/2)}{\cancel{2}\cancel{\sin(\frac{x}{2})}\cdot\cos(\frac{x}{2})} = \frac{\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^3\frac{1}{\lfloor 3} + \left(\frac{x}{2}\right)^5\frac{1}{\lfloor 5} - \cdots}{1 - \left(\frac{x}{2}\right)^2\frac{1}{\lfloor 2} + \left(\frac{x}{2}\right)^4\frac{1}{\lfloor 4} - \cdots}$$

$$= \left(\frac{x}{2} - \frac{x^3}{8\lfloor 3} + \frac{x^5}{32\lfloor 5} - \cdots\right)\left[1 - \left\{\frac{x^2}{4\lfloor 2} - \frac{x^4}{16\lfloor 4} + \cdots\right\}\right]^{-1}$$

$$= \left(\frac{x}{2} - \frac{x^3}{8\lfloor 3} + \frac{x^5}{32\lfloor 5} - \cdots\right)\left[1 + \left\{\frac{x^2}{4\lfloor 2} - \frac{x^4}{16\lfloor 4} + \cdots\right\} + \left\{\frac{x^2}{4\lfloor 2} - \frac{x^4}{16\lfloor 4} + \cdots\right\}^2 + \cdots\right]$$

$$= \frac{x}{2} - \frac{x^3}{8\lfloor 3} + \frac{x^5}{32\lfloor 5} + \frac{x^3}{2\cdot 4\cdot\lfloor 2} - \frac{x^5}{2\cdot 16\cdot\lfloor 4} - \frac{x^5}{8\lfloor 3\,\lfloor 2\cdot 4} + \frac{x^5}{2(4\lfloor 2)^2} - \cdots$$

$$= \frac{x}{2} + \left(\frac{-1}{8\lfloor 3} + \frac{1}{8\lfloor 2}\right)x^3 + \left(\frac{1}{32\lfloor 5} - \frac{1}{32\lfloor 4} - \frac{1}{32\lfloor 3\lfloor 2} + \frac{1}{32(\lfloor 2)^2}\right)x^5 + \cdots$$

$$= \frac{x}{2} + \frac{1}{8}\cdot\frac{3-1}{6}x^3 + \frac{1}{32}\cdot\frac{1}{\lfloor 5}(1-5-10+30)x^5 + \cdots$$

$$= \frac{x}{2} + \frac{x^2}{24} + \frac{x^5}{240} + \cdots$$

(6) Show that the roots of $8x^3-4x^2-4x+1=0$ are $\cos(\frac{\pi}{7})$, $\cos(\frac{3\pi}{7})$ and $\cos(\frac{5\pi}{7})$ and deduce from it the equation whose roots are $\sec^2(\frac{\pi}{7})$, $\sec^2(\frac{3\pi}{7})$, $\sec^2(\frac{5\pi}{7})$.

Soln:- Let $7\theta = (2n+1)\pi \Rightarrow \theta = (2n+1)\frac{\pi}{7}$.

$\therefore \cos\theta = \cos(2n+1)\frac{\pi}{7}$, $n = 0, 1, 2, 3, 4, 5, 6$

$n=0$, $\cos\theta = \cos\frac{\pi}{7}$

$n=1$, $= \cos\frac{3\pi}{7}$

$n=2$, $= \cos\frac{5\pi}{7}$

$n=3$, $= \cos\pi$

$n=4$, $= \cos\frac{9\pi}{7} = \cos(2\pi - \frac{5\pi}{7}) = \cos\frac{5\pi}{7}$

$n=5$, $= \cos\frac{11\pi}{7} = \cos(2\pi - \frac{3\pi}{7}) = \cos\frac{3\pi}{7}$

$n=6$, $= \cos\frac{13\pi}{7} = \cos(2\pi - \frac{\pi}{7}) = \cos\pi/7$

Hence distinct values of $\cos\theta$ are

$\cos\frac{\pi}{7}$, $\cos 3\frac{\pi}{7}$, $\cos\frac{5\pi}{7}$, $\cos\frac{\pi}{7}$, $\cos\pi = (-1)$

Again, $7\theta = (2n+1)\pi$

$\Rightarrow 4\theta + 3\theta = (2n+1)\pi$

$4\theta = (2n+1)\pi - 3\theta$

$\cos 4\theta = \cos\{(2n+1)\pi - 3\theta\} = -\cos 3\theta$

$\Rightarrow \cos 2(2\theta) = -\cos 3\theta$

$\Rightarrow 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow 2(2\cos^2\theta - 1)^2 - 1 = -4\cos^3\theta + 3\cos\theta$

$\Rightarrow 8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$

$\Rightarrow 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$ —— (I) where $x = \cos\theta$.

The roots of this equation in x are -1, $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$.

Since $x = -1$ is a root of (I), $(x+1)$ is a factor of the eqn.

Therefore, dividing the eqn (I) by $(x+1)$, we get the equation (6)

$$8x^3 - 4x^2 - 4x + 1 = 0 \quad \text{——— (II)}$$

whose roots are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$.

Now putting $\frac{1}{x^2} = y$, we get the eqn whose roots are

$\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$, $\sec^2\frac{5\pi}{7}$.

from (II) $\quad 8x^3 - 4x = 4x^2 - 1$

$\Rightarrow \quad 4x(2x^2 - 1) = 4x^2 - 1$

or $\quad 4x\left(\frac{2}{y} - 1\right) = \frac{4}{y} - 1$

or $\quad 4x(2 - y) = 4 - y$

squaring

or $\quad 16x^2(4 + y^2 - 4y) = 16 + y^2 - 8y$

or $\quad \frac{16}{y}(4 + y^2 - 4y) = 16 + y^2 - 8y$

$\therefore \quad y^3 - 24y^2 + 80y - 64 = 0$

7 (i) Prove that the roots of the equation
$8x^3+4x^2-4x-1=0$ are $\cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{4\pi}{7}\right)$ and $\cos\left(\frac{6\pi}{7}\right)$.

(ii) Prove that $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7} = \frac{1}{8}$

(iii) Prove that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

(iv) Evaluate $\sec\left(\frac{2\pi}{7}\right) + \sec\left(\frac{4\pi}{7}\right) + \sec\left(\frac{6\pi}{7}\right)$.

Soln: Let $7\theta = 2n\pi$

$\Rightarrow 4\theta = 2n\pi - 3\theta$

$\cos 4\theta = \cos(2n\pi - 3\theta) = \cos 3\theta$

$\Rightarrow 2(2\cos^2\cancel{\theta}-1)^2 - 1 = 4\cos^3\theta - 3\cos\theta$

Let $\cos\theta = x$

$\Rightarrow 2(2x^2-1)^2 - 1 = 4x^3 - 3x$

$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0$ — (I)

$\because \theta = \frac{2n\pi}{7} \Rightarrow \cos\theta = \cos\frac{2n\pi}{7}$

All the different values of θ are obtained by putting
$n = 0, 1, 2, 3$.

$\therefore$ The distinct values of $\cos\theta$ are
$\cos 0 = 1$, $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{6\pi}{7}$.

Since $x=1$ is a root of the eqn (I) therefore $(x-1)$ is a factor of the eqn (I).

$\therefore$ Dividing (I) by $(x-1)$, $8x^3+4x^2-4x-1=0$ — (II)

whose roots are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{6\pi}{7}$.

(ii) $\cos\frac{2\pi}{7} \cdot \cos\frac{4\pi}{7} \cdot \cos\frac{6\pi}{7} = \frac{1}{8}$

(iii) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{4}{8} = -\frac{1}{2}$.

(iv) Put $y = \frac{1}{x} = \sec\frac{2\pi}{7}$. Then

$y^3 + 4x^2 - 4y - 8 = 0$ is the eqn whose roots are
$\sec\frac{2\pi}{7}$, $\sec\frac{4\pi}{7}$, $\sec\frac{6\pi}{7}$

$\therefore \sec\frac{2\pi}{7} + \sec\frac{4\pi}{7} + \sec\frac{6\pi}{7} = -4$

(8) Find the value of

$$\sec\left(\frac{2\pi}{9}\right) + \sec\left(\frac{4\pi}{9}\right) + \sec\left(\frac{6\pi}{9}\right) + \sec\left(\frac{8\pi}{9}\right).$$

Soln: Let $9\theta = 2n\pi \Rightarrow 5\theta = 2n\pi - 4\theta$

$\cos 5\theta = \cos(2n\pi - 4\theta) = \cos 4\theta = \cos 2(2\theta) = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1$

or $\cos 5\theta + \cos\theta - \cos\theta = 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$

or $2\cos\left(\frac{5\theta+\theta}{2}\right)\cdot\cos\left(\frac{5\theta-\theta}{2}\right) - \cos\theta = 8x^4 - 8x^2 + 1$ where $x = \cos\theta$

or $2\cos 3\theta \cdot \cos 2\theta - \cos\theta = 8x^4 - 8x^2 + 1$

or $2(4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - \cos\theta = 8x^4 - 8x^2 + 1$

or $2(4x^3 - 3x)(2x^2 - 1) - x = 8x^4 - 8x^2 + 1$

$\Rightarrow (x-1)(16x^4 + 8x^3 - 12x^2 - 4x + 1) = 0$

Now $x - 1 = 0 \Rightarrow x = 1$ corresponds to $\cos\theta = 1 = \cos 0$
$\Rightarrow \theta = 0$

Hence the other roots are

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0$$

The roots of this eqn are $\cos\frac{2n\pi}{9}$, $n = 1, 2, 3, 4$.

Now put $x = \frac{1}{y}$; then

$$y^4 - 4y^3 - 12y^2 + 8y + 16 = 0.$$

and the roots of this equation are

$$\sec\left(\frac{2n\pi}{9}\right), \quad n = 1, 2, 3, 4.$$

$$\therefore \sec\left(\frac{2\pi}{9}\right) + \sec\left(\frac{4\pi}{9}\right) + \sec\left(\frac{6\pi}{9}\right) + \sec\left(\frac{8\pi}{9}\right) = 4$$

① Prove that $\sin^{-1}(i) = 2n\pi - i\log(\sqrt{2}-1)$

Sol^n:- Let $\sin^{-1}(i) = \theta \Rightarrow \sin\theta = i$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-i^2} = \sqrt{1+1} = \sqrt{2}$$

Now $\log(\sqrt{2}-1) = \log(\sqrt{2}+i^2) = \log(\cos\theta + i\sin\theta) = \log e^{i\theta}$

$$= \log(e^{i\theta}.1) = \log(e^{i\theta}.e^{2n\pi i}) = \log(e^{i(\theta+2n\pi)})$$

$$= i(\theta + 2n\pi)$$

$$-i^2\log(\sqrt{2}-1) = \cancel{i}(\theta+2n\pi) \Rightarrow \sin\theta = 2n\pi - i\log(\sqrt{2}-1)$$

② Prove that $\sin(\log i^i) = -1$

$$L.H.S = \sin(\log i^i) = \sin(i\log i) = \sin\left(i\log\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right)\right) = \sin\left(i\log e^{i\frac{\pi}{2}}\right)$$

$$= \sin\left(i^2\tfrac{\pi}{2}\right) = \sin\left(-\tfrac{\pi}{2}\right) = -\sin\tfrac{\pi}{2} = -1.$$

③ If $\tan(x+iy) = u+iv$, prove that $u^2+v^2+2u\cot 2x = 1$

Sol^n:- Given, $\tan(x+iy) = u+iv$

$$\Rightarrow x+iy = \tan^{-1}(u+iv)$$

Replacing i by $-i$ $\Rightarrow x-iy = \tan^{-1}(u-iv)$

Now $2x = (x+iy)+(x-iy)$

$$= \tan^{-1}(u+iv) + \tan^{-1}(u-iv)$$

$$= \tan^{-1}\left(\frac{(u+iv)+(u-iv)}{1-(u+iv)(u-iv)}\right)$$

$$= \tan^{-1}\left(\frac{2u}{1-u^2-i^2v^2}\right) = \tan^{-1}\left(\frac{2u}{1-u^2+v^2}\right)$$

$$\tan 2x = \frac{2u}{1-u^2+v^2} \Rightarrow \cot 2x = \frac{1-u^2+v^2}{2u}$$

$$\therefore u^2+v^2+2u\cot 2x = 1$$

④ If $\tan\log(x+iy) = a+ib$, when $a^2+b^2 \neq 1$, prove that

$$\tan\{\log(x^2+y^2)\} = \frac{2a}{1-a^2-b^2}$$

Sol^n:- $\tan\{\log(x+iy)\} = a+ib$

Replacing i by $-i$: $\tan\{\log(x-iy)\} = a-ib$

Now $\tan\{\log(x^2+y^2)\} = \tan\{\log(x^2-i^2y^2)\}$

$$= \tan\{\log(x+iy)(x-iy)\}$$

$$= \tan\{\underbrace{\log(x+iy)}_{A} + \underbrace{\log(x-iy)}_{B}\}$$

$$= \frac{\tan A + \tan B}{1-\tan A.\tan B} = \frac{\tan\{\log(x+iy)\} + \tan\{\log(x-iy)\}}{1-\tan\{\log(x+iy)\}.\tan\{\log(x-iy)\}}$$

$$= \frac{2a}{1-a^2-b^2}$$

(5) If $i^{\alpha+i\beta} = \alpha+i\beta$, prove that $\alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$.

Sol^n:- $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

Now $i^{\alpha+i\beta} = \left(e^{i\frac{\pi}{2}}\right)^{\alpha+i\beta} = \alpha+i\beta$ (given)

$$\left(e^{i\frac{\pi}{2}}\cdot 1\right)^{\alpha+i\beta} = \alpha+i\beta$$

$$\left(e^{i\frac{\pi}{2}}\cdot e^{2n\pi i}\right)^{\alpha+i\beta} = \alpha+i\beta$$

$$\Rightarrow e^{i\left(\frac{\pi}{2}+2n\pi\right)(\alpha+i\beta)} = \alpha+i\beta$$

$$\Rightarrow e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\cdot e^{i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha+i\beta$$

or $e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\cdot\left[\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha + i\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha\right] = \alpha+i\beta$

Equating real and imaginary parts

$e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\cdot\cos\left\{\left(2n\pi+\frac{\pi}{2}\right)\alpha\right\} = \alpha$ and $e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\cdot\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha = \beta$.

Squaring & adding $\alpha^2+\beta^2 = \left\{e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\right\}^2 = e^{-(4n+1)\pi\beta}$

(6) Express $\log\{\log(\cos\theta + i\sin\theta)\}$ in the form $A+iB$, where A and B are real.

Sol^n:- Expression $= \log\{\log e^{i\theta}\} = \log\{i\theta\log e\} = \log\{i\theta\cdot 1\}$

$= \log(i\theta) + \log 1 = \log i + \log\theta + 0 = \log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) + \log\theta$

$= \log e^{i\frac{\pi}{2}} + \log\theta = i\frac{\pi}{2}\cdot\log e + \log\theta = i\frac{\pi}{2} + \log\theta$

$= A + iB$

where $A = \log\theta$, $B = \frac{\pi}{2}$.

(7) If α, β be the imaginary cube roots of unity, prove that (11)

$$\alpha e^{\alpha x} + \beta e^{\beta x} = -e^{-\frac{x}{2}}\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + \cos\frac{\sqrt{3}}{2}x\right]$$

Soln: Imaginary cube roots of unity are $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$.

Let $\alpha = \frac{-1+i\sqrt{3}}{2} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\beta = \frac{-1-i\sqrt{3}}{2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$L.H.S = \alpha e^{\alpha x} + \beta e^{\beta x} = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) e^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}$$

$$= -\frac{1}{2} e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} + i\frac{\sqrt{3}}{2} e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} - \frac{1}{2} e^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x} - i\frac{\sqrt{3}}{2} e^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}$$

$$= -\frac{1}{2} e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right) + i\frac{\sqrt{3}}{2} e^{-\frac{x}{2}}\cdot\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right)$$
$$-\frac{1}{2} e^{-\frac{x}{2}}\cdot\left(\cos\frac{\sqrt{3}}{2}x - i\sin\frac{\sqrt{3}}{2}x\right) - i\frac{\sqrt{3}}{2} e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x - i\sin\frac{\sqrt{3}}{2}x\right)$$

$$= -2\cdot\frac{1}{2} e^{-\frac{x}{2}}\cdot\cos\frac{\sqrt{3}}{2}x + \cancel{2}i^2\frac{\sqrt{3}}{\cancel{2}} e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$$

$$= -e^{-\frac{x}{2}}\cdot\cos\frac{\sqrt{3}}{2}x - \sqrt{3}\, e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$$

(8) If $i^{i^{i\cdots\infty}} = A + iB$, prove that

$$\tan\left(\frac{\pi A}{2}\right) = \frac{B}{A} \text{ and } A^2 + B^2 = e^{-\pi B}.$$

Soln:

$$i^{i^{i\cdots\infty}} = A + iB \Rightarrow i^{A+iB} = A + iB \Rightarrow \left(e^{i\frac{\pi}{2}}\right)^{A+iB} = A + iB$$

$$\Rightarrow e^{i\frac{\pi}{2}A}\cdot e^{-\frac{\pi}{2}B} = A + iB \Rightarrow \left(\cos\frac{\pi}{2}A + i\sin\frac{\pi}{2}A\right)e^{-\frac{\pi}{2}B} = A + iB$$

Equating: $A = e^{-\frac{\pi}{2}B}\cdot\cos\frac{\pi}{2}A$ and $B = e^{-\frac{\pi}{2}B}\sin\frac{\pi A}{2}$.

squaring & adding $A^2 + B^2 = \left(e^{-\frac{\pi}{2}B}\right)^2\cdot\left\{\cos^2\frac{\pi A}{2} + \sin^2\frac{\pi A}{2}\right\} = e^{-\pi B}$ Ans

Dividing $\frac{B}{A} = \frac{\sin \pi A/2}{\cos \pi A/2}$

$\therefore \tan\left(\frac{\pi A}{2}\right) = \frac{B}{A}$ Ans

(9) If $\tan(x+iy) = u+iv$, prove that $u^2+v^2+2u\cot 2x = 1$.

(12)

Sol^n:- since $\tan(x+iy) = u+iv \Rightarrow \tan^{-1}(u+iv) = x+iy$

Replacing i by $-i$ $\Rightarrow \tan^{-1}(u-iv) = x-iy$

Now $2x = (x+iy)+(x-iy)$

$= \tan^{-1}(u+iv) + \tan^{-1}(u-iv)$

$= \tan^{-1} \dfrac{(u+iv)+(u-iv)}{1-(u+iv)(u-iv)} = \tan^{-1} \dfrac{2u}{1-(u^2-i^2v^2)}$

$= \tan^{-1}\left(\dfrac{2u}{1-u^2-v^2}\right)$

or $\dfrac{\tan 2x}{1} = \dfrac{2u}{1-u^2-v^2} \Rightarrow \dfrac{\cot 2x}{1} = \dfrac{1-u^2-v^2}{2u}$

$\therefore u^2+v^2+2u\cot 2x = 1$ P.

(10) Prove that $i \log\left(\dfrac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$

Sol^n:- Put $\left.\begin{array}{l} x = r\cos\theta \\ 1 = r\sin\theta \end{array}\right\} \Rightarrow r^2 = 1+x^2 \Rightarrow r = \sqrt{1+x^2}$

$\tan\theta = \frac{1}{x} \Rightarrow x = \cot\theta$

L.H.S $= i\log\left(\dfrac{x-i}{x+i}\right) = i\log\left(\dfrac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right) = i\log e^{-2i\theta}$

$= -2i^2\theta \log e = 2\theta = 2\cot^{-1}(x) = 2\left(\frac{\pi}{2} - \tan^{-1}x\right)$

$= \pi - 2\tan^{-1}x$ P.

13

1) (a) Determine the constants A, B (in terms of θ) so that the relation $\frac{\cos\theta - x}{1-2x\cos\theta + x^2} = \frac{A}{1-xe^{i\theta}} + \frac{B}{1-xe^{-i\theta}}$

may hold for all values of x.

(b) Hence or otherwise expand $\frac{\cos\theta - x}{1-2x\cos\theta+x^2}$ in the form $\sum_{n=0}^{\infty} a_n x^n$, where a_n is a function of θ, which you are asked to Calculate.

Also state the conditions of Validity.

(c) Expand $\frac{x\cos\theta - x^2}{1-2x\cos\theta+x^2}$ in a series of ascending powers of x, where $-1 < x < 1$

Soln: (a) $\because$

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

Given expression =

$$\frac{\cos\theta - x}{1-2x\cos\theta+x^2} = \frac{\frac{e^{i\theta}+e^{-i\theta}}{2} - x}{1-x(e^{i\theta}+e^{-i\theta})+x^2} = \frac{\frac{1}{2}(e^{i\theta}+e^{-i\theta}-2x)}{1-xe^{i\theta}-xe^{-i\theta}+x^2}$$

$$= \frac{1}{2}\,\frac{(e^{i\theta}-x)+(e^{-i\theta}-x)}{(1-xe^{i\theta}) - xe^{-i\theta}(1-xe^{i\theta})} = \frac{1}{2}\left(\frac{e^{i\theta}}{1-xe^{i\theta}} + \frac{e^{-i\theta}}{1-xe^{-i\theta}}\right)$$

$$= \frac{\frac{1}{2}e^{i\theta}}{1-xe^{i\theta}} + \frac{\frac{1}{2}e^{-i\theta}}{1-xe^{-i\theta}}$$

Hence $A = \frac{1}{2}e^{i\theta}$

$B = \frac{1}{2}e^{-i\theta}$

(b) $\frac{\frac{1}{2}e^{i\theta}}{1-xe^{i\theta}} + \frac{\frac{1}{2}e^{-i\theta}}{1-xe^{-i\theta}}$

$$= \frac{1}{2}e^{i\theta}(1-xe^{i\theta})^{-1} + \frac{1}{2}e^{-i\theta}(1-xe^{-i\theta})^{-1}$$

$$= \frac{1}{2}\left[e^{i\theta}\left[1+xe^{i\theta}+x^2e^{2i\theta}+\dots \text{to}\infty\right) + e^{-i\theta}\left(1+xe^{-i\theta}+x^2e^{-2i\theta}+\dots\infty\right)\right]$$

This expansion is valid if $|xe^{i\theta}| < 1$ and $|xe^{-i\theta}| < 1$

$\Rightarrow |x||e^{i\theta}| < 1$ and $|x||e^{-i\theta}| < 1$

$\therefore$ Given expression

$$= \frac{1}{2}\left[(e^{i\theta}+e^{-i\theta}) + x(e^{i\theta}+e^{-i\theta}) + x^2(e^{2i\theta}+e^{-2i\theta}) + \dots \text{to}\infty\right]$$

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$= \frac{1}{2}\left[2\cos\theta + 2x\cos 2\theta + 2x^2\cos 3\theta + \cdots \infty\right]$

$= \cos\theta + x\cos 2\theta + x^2\cos 3\theta + \cdots$ to ∞

$= \sum_{n=0}^{\infty} x^n \cos(n+1)\theta$ which is of the form $\sum_{n=0}^{\infty} x^n a_n$.

Evidently $a_n = \cos(n+1)\theta$.

(c) From (b)

$$\frac{\cos\theta - x}{1 - 2x\cos\theta + x^2} = \cos\theta + x\cos 2\theta + x^2\cos 3\theta + \cdots \text{ to } \infty$$

$$\therefore \frac{x\cos\theta - x^2}{1 - 2x\cos\theta + x^2} = x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \cdots \text{ to } \infty$$

2) (a) Expand $\frac{x\sin\theta}{1 - 2x\cos\theta + x^2}$ in powers of x.

(b) If $|x| < 1$, prove that $\frac{\sin\theta}{1 - 2x\cos\theta + x^2} = \sum_{n=1}^{\infty} x^{n-1} \sin n\theta$.

Degree - I

TRIGONOMETRY
DEMOIVRE'S THEOREM

ORIENTAL COLLEGE, PATNA CITY, PATNA

T: 1 hr

Prof (DR). MD. EQBALUZAFAR

F.M. 40

1. Prove that $\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$

2. If $x + \frac{1}{x} = 2\cos\alpha$, $y + \frac{1}{y} = 2\cos\beta$, $z + \frac{1}{z} = 2\cos\gamma, \ldots$ prove that one of the values of $xyz\ldots + \frac{1}{xyz\ldots} = 2\cos(\alpha+\beta+\gamma+\ldots)$

3. Prove that one value of $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}}$ is $2(a^2+b^2)^{\frac{m}{2n}}\cos\left(\frac{m}{n}\tan^{-1} b/a\right)$.

4. Find a value of x such that
$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{\sin n\theta}{\sin^n\theta}$$
where α and β are the roots of $t^2 - 2t + 2 = 0$.

5. If α and β are the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

6. If $\sin x + \sin y + \sin z = 0$, $\cos x + \cos y + \cos z = 0$, Show that $\sin 2x + \sin 2y + \sin 2z = 0 = \cos 2x + \cos 2y + \cos 2z$.

7. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, prove that $\sum \sin 2\alpha = \sum \cos 2\alpha = 0$ and $\sum \sin^2\alpha = \sum \cos^2\alpha = \frac{3}{2}$

8. If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$, prove that
$$\cos n\alpha + \cos n\beta + \cos n\gamma = 3\cos\frac{n}{3}(\alpha+\beta+\gamma) \text{ or } 0$$
according as n is or not a multiple of 3.

—.—